Competitive Outcomes and the Inner Core of NTU Market Games

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Abstract

We investigate the relationship between subsets of the inner core for NTU market games and competitive payoff vectors of certain markets linked to the NTU market game. This can be considered as the case in between the two extreme cases of Qin [3]. We extend his results to a large class of closed subsets of the inner core: Given an NTU market game we construct a market depending on a given closed subset of the inner core. This market represents the game and further has the given set as the set of payoffs of competitive equilibria. It turns out that this market is not determined uniquely and thus we obtain a class of markets with the desired property.

1. Introduction

The idea to consider cooperative games as economies or markets goes back to Shapley and Shubik [4]. They look at TU market games. These are cooperative games with transferable utility (TU) that are in a certain sense linked to economies or markets. More precisely, a market represents a game if the set of utility allocations a coalition can reach in the market coincides with the set of utility allocations a coalition obtains according to the coalitional function of the game. If there exists a market which represents a game, then this game is called a market game.

Cooperative games with non-transferable utility (NTU) can be considered as a generalization of TU games, where the transfer of the utility within a coalition does not take place at a fixed rate. Here, we consider NTU market games. Billera and Bixby [1] show that every totally balanced NTU game, that is compactly convexly generated, is an NTU market game. The inner core is a refinement of the core for NTU games. A point is in the inner core if is in the core and furthermore there exists a transfer rate vector, such that - given this transfer rate vector - no coalition can improve even if utility can be transferred within a coalition according to this vector. So, an inner core point is in the core of an associated hyperplane game where the utility can be transferred according to the transfer rate vector. Qin [3] shows, following a conjecture of Shapley and Shubik [5], that on the one hand there exists a market that has the complete inner core as its set of competitive payoff vectors and that on the other hand there is a market that has a given inner core point as its unique competitive payoff vector.

Here, we investigate the case in between these two extreme cases of Qin [3]. We extend his results to a large class of closed subsets of the inner core: Given an NTU market game we construct a market depending on a given closed subset of the inner core. This market represents the game and further has the given set as the set of payoffs of competitive equilibria. It turns out that this market is not determined uniquely. Several parameters in our construction can be chosen in different ways. Thus, we obtain a class of markets with the desired property.

2. Main Results

Let (N,V) be a compactly, convexly generated and totally balanced NTU game and A be a compact subset of its inner core, such that the following condition is satisfied:

There exists a mapping $\lambda : A \to \Delta_{++}$ and an $\varepsilon > 0$, that associates to every point $a \in A$ a multiplier $\lambda(a) = \lambda^a$, such that every point $a \in A$ can be strictly separated from the set $V(N)\setminus\{a\}$ using this normal vector λ^a, i.e.

$$\lambda^a \cdot a > \lambda^a \cdot x \quad \text{for all } x \in V(N) \setminus \{a\},$$

and for all $a \in A$ the normal vector λ^a is strictly greater than ε in every coordinate, i.e.

$$\lambda_i^a > \varepsilon \quad \text{for all } i \in N.$$

Theorem. *There exists a market such that this market represents the game (N,V) and such that the set of competitive payoff vectors of this market is the set A.*

To show this theorem we construct a market, which satisfies all the desired properties. We call it the *induced A-market*. In the following we define this market.

We make use of an auxiliary game $(N,\tilde{V})$ that we derive from (N,V). Let $(N,\tilde{V})$ be the NTU-game defined by

$$\tilde{V}(S) = \begin{cases} V(S) & \text{if } S \subset N \\ \bigcap_{a\in A} \{x \in \mathbb{R}^n | \lambda^a \cdot x \leq \lambda^a \cdot a\} & \text{if } S = N \end{cases}$$

where λ^a is the normal vector from above.

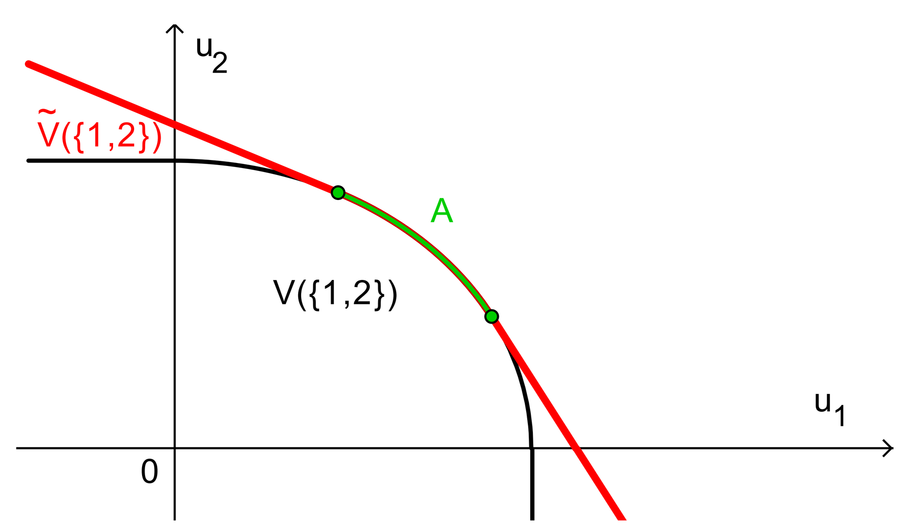

Figure 1: *An example for the sets $V(\{1,2\})$ and $\tilde{V}(\{1,2\})$ for $N = \{1,2\}$*

The games (N,V) and $(N,\tilde{V})$ are equal except for the grand coalition N. For the coalition N we extend the set $\tilde{V}(N)$ depending on the normal vectors of the set A.

Now let $x \in \tilde{V}(N)$ and

$$\bar{t}^x = \min\left\{t \in \mathbb{R}_+ | x - te^N \in V(N)\right\}.$$

Define the mapping P_A between the games $(N,\tilde{V})$ and (N,V) by

$$P_A : \tilde{V}(N) \longrightarrow V(N)$$

via

$$P_A(x) = x - \bar{t}^x e^N.$$

Note, that if $x \in V(N)$ then $\bar{t}^x = 0$ and $P_A(x) = x$. The mapping P_A is continuous and its image is $V(N)$. Define

$$\tilde{C}^N = \left\{z \in \tilde{V}(N) | P_A(z) \in C^N\right\}.$$

Then we have $P_A\left(\tilde{C}^N\right) = C^N$.

To simplify the notation of the market, we introduce some sets before:

For the definition of the production sets define for all non-empty coalitions $S \subset N$

$$A_S^1 = \left\{\left(c^S, -e^S, c^S, -e^S, -e^S\right) | c^S \in C^S\right\},$$
$$A_S^2 = \left\{\left(c^S, 0, c^S, -e^S, 0\right) | c^S \in C^S\right\},$$
$$A_S^3 = \left\{\left(c^S, 0, c^S, 0, -e^S\right) | c^S \in C^S\right\},$$

and for the grand coalition N define

$$A_N^1 = \left\{\left(P_A\left(\tilde{c}^N\right), -e^N, \tilde{c}^N, -e^N, -e^N\right) | \tilde{c}^N \in \tilde{C}^N\right\},$$
$$A_N^2 = \left\{\left(P_A\left(\tilde{c}^N\right), 0, \tilde{c}^N, -e^N, 0\right) | \tilde{c}^N \in \tilde{C}^N\right\},$$
$$A_N^3 = \left\{\left(P_A\left(\tilde{c}^N\right), 0, \tilde{c}^N, 0, -e^N\right) | \tilde{c}^N \in \tilde{C}^N\right\}.$$

The *induced A-market* of the game (N,V) and the given set A is defined by

$$\mathcal{E}_{V,A} = (X^i, Y^i, u^i, \omega^i)_{i\in N}$$

with for every individual $i \in N$

- $X^i = \mathbb{R}_+^n \times \{0\} \times \mathbb{R}_+^n \times \{0\} \times \{0\} \subseteq \mathbb{R}^{5n}$,
- $Y^i = convexcone\left[\bigcup_{S\subseteq N}(A_S^1 \cup A_S^2 \cup A_S^3)\right] \subseteq \mathbb{R}^{5n}$,
- $\omega^i = (0, e^{\{i\}}, 0, e^{\{i\}}, e^{\{i\}})$,
- $u^i : X^i \to \mathbb{R}$ with

$$u^i(x^{(1)}, 0, x^{(3)}, 0, 0) = \min\left(x_i^{(1)}, x_i^{(3)} + \varepsilon \sum_{j\neq i} z_j^{(3)}\right).$$

3. Conclusion

In this paper we continue the work of Shapley and Shubik [5] and Qin [3] to investigate the relationship between competitive payoffs of markets, that represent a cooperative game, and their relation to solution concepts for cooperative games. We extend the results of Qin [3] to a large class of closed subsets of the inner core.

Having our result in mind there remains the open question if we can weaken further our assumptions such that the results can be proven for more general cases. Another interesting related line of research is to continue to look at the class of games that are linked to coalition production economies as analyzed by Inoue [2]. Moreover, it is interesting to compare the set of competitive equilibrium allocations of different market representations of a given NTU market game. Does there exist a general and more simple method to obtain desired competitive payoffs? Can we characterize a class of NTU games where this is possible? What happens if we restrict our attention for example to bargaining games?

References

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