

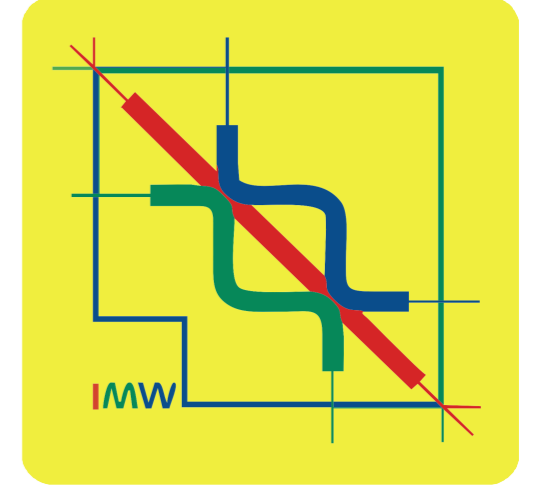
Indivisible Commodities and an Equivalence Theorem

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on the Strong Core

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EBIM – Economic Behavior and Interaction Models



1. Introduction

In the literature, all commodities are usually assumed to be perfectly divisible for the convenience of the analysis. Perfect divisibility is one extreme assumption as a description of the real world. We analyze the opposite extreme case where all commodities can be consumed only in integer quantities. The indivisibility of commodities makes consumers' consumption sets non-convex. Although the non-convexity is thought as an obstacle, we prove an equivalence between the core and Walras equilibria.

A Walras equilibrium is an equilibrium state in market places. All consumers face the same prices, they regard the prices as given, and they try to maximize their utilities under budget constraints. A Walras equilibrium is a state where, under some prices, the total demand for each commodity is equal to the total amount of the commodity available in the market. In contrast, the core is an institution-free concept and it is the set of all stable states in the sense that any group of consumers cannot make benefits by reallocating their endowments among themselves. In particular, the core is defined independently of prices.

Debreu and Scarf [1] considered a sequence of replica economies with divisible commodities. The n th replica economy consists of n replica consumers for each consumer of the original economy. By virtue of the equal treatment property of core allocations, consumers who are replicas of the same consumer are allocated the same consumption bundle at any core allocations. This enables us to regard core allocations of replica economies as having the same dimension as Walras allocations of the original economy. If we replicate economy more, the number of possible groups of consumers becomes large and, therefore, more allocations are improved upon. Hence, the sequence of cores is shrinking. Debreu and Scarf [1] proved that the limit of the decreasing sequence of cores of replica economies coincides with Walras equilibria of the original economy. Thus, if the number of consumers is large, at any core allocation, consumers exchange their endowments as if they would face a common price vector. Therefore, this core equivalence theorem explains the emergence of prices. In the case of divisible commodities, however, infinitely many consumers are needed to hold the core equivalence.

In our model with indivisible commodities, by contrast, even in an economy with finitely many consumers, a similar core equivalence can be obtained.

2. Model

An economy \mathcal{E} is defined as a mapping from the set A of consumers to the space of consumers' characteristics. All consumers are assumed to have the same consumption set \mathbb{Z}_+^L where $L \in \mathbb{Z}$ ($L \geq 2$) is the number of indivisible commodities. Every consumer $a \in A$ is characterized by his preference relation \succsim_a on \mathbb{Z}_+^L and his endowment vector $e(a) \in \mathbb{Z}_+^L$. Consumers' preference relations are assumed to be reflexive, transitive, complete, and weakly monotone [if $x, y \in \mathbb{Z}_+^L$ and $x \geq y$, then $x \succsim y$].

Definition. An allocation $f : A \rightarrow \mathbb{Z}_+^L$ is a *strong core allocation* of economy \mathcal{E} if it is feasible, i.e.,

$\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$ and there exists no coalition $S \subseteq A$ and allocation $g : S \rightarrow \mathbb{Z}_+^L$ within S such that $\sum_{a \in S} g(a) = \sum_{a \in S} e(a)$, $g(a) \succsim_a f(a)$ for all $a \in S$, and $g(a) \succ_a f(a)$ for at least one $a \in S$. The strong core $C_S(\mathcal{E})$ of economy \mathcal{E} is the set of all strong core allocations of economy \mathcal{E} .

The size of the core depends on the improvement defining the core. Here, we adopt the same improvement that defines Pareto efficiency. When we adopt a stronger improvement which requires that all consumers in a coalition can be strictly better off, the resulting core is called the *weak core*. By definition, the strong core is a subset of the weak core.

If commodities are perfectly divisible and if consumers' preference relations are continuous and strongly monotone, then, by transfer of a small amount of any commodity from the consumer who is made strictly better off to the other consumers in the coalition, all consumers in the coalition can be strictly better off. Therefore, in this case, the strong core coincides with the weak core.

In our economy with indivisible commodities, however, the strong core can be strictly smaller than the weak core because a small amount of commodity transfer is impossible. In addition, even the weak core can be empty (Inoue [3] gave a sufficient condition for the weak core to be nonempty, but any sufficient condition for the nonemptiness of the strong core has not been obtained).

Definition. A pair (p, f) of a price vector $p \in \mathbb{Q}_+^L$ and a feasible allocation $f : A \rightarrow \mathbb{Z}_+^L$ is a *cost-minimized Walras equilibrium* of economy \mathcal{E} if

- (i) for all $a \in A$, $p \cdot f(a) \leq p \cdot e(a)$,
- (ii) for all $a \in A$, if $x \in \mathbb{Z}_+^L$ and $x \succ_a f(a)$, then $p \cdot x > p \cdot e(a)$, and
- (iii) for all $a \in A$, if $x \in \mathbb{Z}_+^L$ and $x \succsim_a f(a)$, then $p \cdot x \geq p \cdot e(a)$.

The set of all cost-minimized Walras allocations of economy \mathcal{E} is denoted by $W_{CM}(\mathcal{E})$.

When a pair (p, f) of a price vector p and a feasible allocation f satisfies the conditions (i) and (ii) above, it is called a *Walras equilibrium*. If consumers' preference relations are locally non-satiated, then the utility maximization (condition (ii)) implies the cost minimization (condition (iii)). In particular, in economies analyzed by Debreu and Scarf [1], cost-minimized Walras equilibria coincides with Walras equilibria. In our economy, since all commodities are indivisible, consumers' preference relations are necessarily locally satiated and there can exist a Walras equilibrium that is not a cost-minimized Walras equilibrium.

By an argument similar to the first welfare theorem, it can be shown that any cost-minimized Walras allocation is a strong core allocation, i.e., for every economy \mathcal{E} , $W_{CM}(\mathcal{E}) \subseteq C_S(\mathcal{E})$.

In our main theorem below, we put the further assumption on consumers' preference relations. Let $k \in \mathbb{Z}$ with $k \geq 2$. The set \mathcal{P}_k is a family of preference relations \succsim such that for any distinct commodities h and i and any consumption vector $x \in \mathbb{Z}_+^L$, if $x^{(i)} \geq 1$, then $x - \chi_i + k\chi_h \succ x$, where χ_i is the i th unit vector. Thus, any consumer with preference relation $\succsim \in \mathcal{P}_k$ is willing to give up one unit of a commodity in exchange for k units of another commodity. All preference relations in \mathcal{P}_k have uniformly positive marginal rates of substi-

tution. In particular, the lexicographic ordering is excluded.

A finite subset T of the space $\mathcal{P}_k \times \mathbb{Z}_+^L$ of consumers' characteristics is called a *type set*. In our main theorem, we consider an economy in which every type has a sufficiently large number of consumers. For type $t \in T$, let $t = (\succsim_t, e_t)$ and $A_t = \{a \in A \mid (\succsim_a, e(a)) = t\}$, that is, \succsim_t is the preference relation of consumers of type t , e_t is the endowment vector of consumers of type t , and A_t is the set of consumers of type t .

3. Main Theorem

Theorem. For every $r \geq 1$, every $k \in \mathbb{Z}$ with $k \geq 2$, every type set $T \subseteq \mathcal{P}_k \times \mathbb{Z}_+^L$ with $\#T \leq r$ and $\sum_{t \in T} e_t \gg 0$, there exists $N \in \mathbb{Z}_{++}$ such that, if economy $\mathcal{E} : A \rightarrow T$ satisfies $\#A \geq N$ and $\#A_t/\#A \geq 1/r$ for every type $t \in T$, then

- (1) for every $f \in C_S(\mathcal{E})$, every type $t \in T$, and every consumer $a, b \in A_t$, $f(a) \sim_t f(b)$ holds, and
- (2) $C_S(\mathcal{E}) = W_{CM}(\mathcal{E})$.

If $r = \#T$ in the theorem, then economy \mathcal{E} is a replica economy, that is, every type has the same number $\#A/\#T$ of consumers. Hence, our theorem covers more general economies than replica economies.

From the conditions $\#A \geq N$ and $\#A_t/\#A \geq 1/r$ for every $t \in T$, we require that each type has a sufficiently large number of consumers.

The first result (1) is called *equal treatment property* of strong core allocations. By relying on convexity of preference relations, Debreu and Scarf [1] proved strong core allocations of replica economies with divisible commodities have the equal treatment property. As Green [2] pointed out, this property does not hold in almost all non-replica economies with divisible commodities, i.e., if economy is non-replica one, there can exist a strong core allocation which does not have the equal treatment property. In contrast, in our theorem, consumers' preference relations cannot be convex and economy may be non-replica one, but strong core allocations have the equal treatment property. Our proof of the equal treatment property is quite different from that of Debreu and Scarf [1].

Another different point from the case of divisible commodities is that even in a finite economy, we can obtain the equivalence between the strong core and cost-minimized Walras equilibria. To hold the equivalence in our economy, however, the number of consumers must be sufficiently large. If the size of economy is small, there can exist a strong core allocation that is not a cost-minimized Walras allocation.

A weakness of the above theorem is that, in some economies, the strong core $C_S(\mathcal{E})$ and the set $W_{CM}(\mathcal{E})$ of cost-minimized Walras allocations can be both empty.

References

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