Irreversible Investment Games

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EBIM — Economic Behavior and Interaction Models

1. Introduction

A commonly modeled friction in the investment opportunities of a firm is the irreversibility of an investment decision. The underlying idea is that the installed capital is too specific to be used for different purposes and thus has no resale value if the economic conditions in the relevant markets decline. If the development of these conditions is uncertain, we find ourselves in the setting of *real options*. Since delaying the investment possibly entails better information on the revenue chances, this option allows to reduce ex-post negative returns. As a consequence, investment is optimally carried out only if it yields a strictly positive expected net present value (NPV). This contrasts the zero-NPV rule when capital can be flexibly rented at a cost. The former finding also applies to the case of sequential investment, as studied in Pindyck (1988), where capital can be expanded as often and in bits as small as desired at a proportional cost. These investment problems are mostly viewed in isolation. However, like for options on financial assets, there are typically several owners of an option on the same underlying real asset. Consider e.g. firms with the possibility to enter some market, or incumbents of an oligopoly that may expand capacity limits. In contrast to purely financial options, the value of the underlying asset is then influenced by the exercise strategy of any option holder. This adds a *strategic* aspect to the investment problem. We expect e.g. that the threat of preemption diminishes the value of the option to wait. Grenadier (2002) derived this effect in a strongly homogeneous case, which can be solved by common option pricing techniques.

The aim of this project is to establish a more general and rigorous theory of strategic sequential irreversible investment. This advance is enabled by a new approach to solve the involved stochastic optimal control problems developed in Bank and Riedel (2001). The present work is also a contribution to the sparse literature on stochastic differential games, while we do not restrict strategies to *rates* of control.

2. The game

Our game is in continuous time. Uncertainty and the information revelation is modeled by a

- *filtered probability space* $(\Omega, \mathscr{F}_\infty, \mathbf{P}, (\mathscr{F}_t)_{t\geq 0})$,

where the filtration satisfies the usual conditions of right continuity and completeness. There are

- $n \in \mathbf{N}$ *players*, resp. firms.

Each firm may choose an accumulated investment process with the investment decision at each moment based on the information provided by the filtration. For the moment, suppose the players are identical, so each player i, $i = 1 \dots n$ has the

- *strategy space*

$$\mathscr{A}^i \triangleq \{Q^i \text{ adapted, nondecreasing, left-continuous process with } Q_0^i = 0 \ \mathbf{P}\text{-a.s.}\}.$$

These strategies are of the *open loop* type, since the investment decision at each moment is not contigent on the actual investment by the opponents. Closed loop strategies at this point still pose severe conceptual and technical difficulties, cf. Back and Paulsen (2009).

Given a combination of strategies $(Q^1 \dots Q^n) \in \mathscr{A}^n$, player i receives the expected

- *payoff*

$$J^i(Q^i|Q^{-i}) \triangleq \mathbf{E}\left[\int_0^\infty \Pi(t, Q_t^i, Q_t^{-i})\, dt - \int_0^\infty k_t\, dQ_t^i\right]$$

$$Q \triangleq \sum_{j=1\dots n} Q^j \qquad Q^{-i} \triangleq Q - Q^i \in \mathscr{A}.$$

$\Pi(\omega, t, q^i, q^{-i})$ is a random revenue stream that depends *further* on the current own and opponent capital levels. k is the discounted proportional cost of investment, nonnegative but possibly random as well.

Apart from technical measurability and integrability conditions, we make the following structural assumption regarding the revenue's dependence on capital.

Assumption.

(i)

$$\Pi_{q^iq^i} \triangleq \frac{\partial^2 \Pi}{(\partial q^i)^2} < 0$$

(ii)

$$\Pi_{q^iq^i} + (n-1)\cdot \Pi_{q^iq^{-i}} \leq 0 \qquad \text{(UC)}$$

Revenue is concave in own installed capital. Condition (UC) is among the weakest known conditions to induce uniqueness of equilibrium in the static Cournot game with payoff Π, cf. Vives (1999).

Our notion of equilibrium inherits the open loop qualification from the strategies.

Equilibrium. $(Q^{*_1} \dots Q^{*_n}) \in \mathscr{A}^n$ is an *open loop investment equilibrium* if Q^{*_i} maximizes $J^i(Q^{*_i}|Q^{*_{-i}})$ over $\mathscr{A}$ for all $i = 1 \dots n$, where $Q^{*_{-i}} = \sum_{j=1\dots n, j\neq i} Q^{*_j}$.

This equilibrium is potentially not subgame perfect, since we do not allow reactions to deviating investment.

3. Equilibrium determination

We first characterize best responses by first order conditions. For this purpose we introduce a gradient that can be interpreted as marginal benefit from investment at some stopping time τ. Since investment is irreversible, marginal benefit is composed of a *stream* of marginal revenue, minus investment unit cost:

$$\nabla J^i(Q^i|Q^{-i})_\tau \triangleq \mathbf{E}\left[\int_\tau^\infty \Pi_{q^i}(t, Q_t^i, Q_t^{-i})\, dt \middle| \mathscr{F}_\tau\right] - k_\tau.$$

The first order condition for a best response is then based on this gradient.

Proposition 1 (FOC)**.** *If our Assumption is satisfied, a control policy $Q^{*_i} \in \mathscr{A}$ maximizes firm i's objective J^i for a given process $Q^{-i} \in \mathscr{A}$ if for all stopping times τ*

$$\nabla J^i(Q^*|Q^{-i})_\tau \leq 0,$$

and

$$\int_0^\infty \nabla J^i(Q^{*_i}|Q^{-i})_t\, dQ_t^{*_i} = 0 \quad \mathbf{P}\text{-a.s.}$$

For a best response, the marginal benefit from investment must be nonpositive, but whenever investment is undertaken, the gradient must be zero.

We construct a nondecreasing solution from the following stochastic representation problem, which turns the first order condition into an *equality*. Under our Assumption, there exists following Bank and El Karoui (2004) a unique optional process L satisfying for all stopping times τ

$$\mathbf{E}\left[\int_\tau^\infty \Pi_{q^i}(t, \sup_{\tau\leq u<t} L_u, (n-1)\cdot \sup_{\tau\leq u<t} L_u)\, dt \middle| \mathscr{F}_\tau\right] = k_\tau.$$

It yields the existence of an open loop equilbrium.

Theorem 2 (Existence)**.** *Under our Assumption, the unique symmetric open loop investment equilibrium is given by*

$$Q_t^{*_i} \triangleq \left(\sup_{0\leq u<t} L_u\right)^+ \quad (t \geq 0)$$

for all $i = 1 \dots n$, where L is the optional process solving the representation problem.

4. Cournot competition

Now suppose firms face an inverse demand function and operate at full capacity. Thus instantaneous revenue is

$$\Pi(\omega, t, q^i, q^{-i}) = e^{-rt} P(X_t(\omega), q^i + q^{-i}) q^i$$

and investment is subject to the same discounting

$$k_t = e^{-rt}.$$

Then, symmetry will naturally arise in equilibrium.

Proposition 3 (Catching up)**.** *Set $n = 2$. Assume that firm i has capital Q_0^i installed before the investment game starts, with $Q_0^1 > Q_0^2$. Then, if $P_q \leq 0$, in any open loop equilibrium, $dQ_s^{*_1} = 0$ as long as $Q_0^1 > Q_0^2 + \int_0^t dQ_s^{*_2}$.*

In equilibrium, only the currently smallest firm(s) invest.

Option value of waiting. Increasing n accelerates equilibrium investment and decreases equilibrium profits. In the limit, as $n \to \infty$, a perfectly competitive equilibrium is attained with zero NPV investment. In the limit, the FOC becomes

$$\mathbf{E}\left[\int_\tau^\infty e^{-rt} P(X_t, Q_t^*)\, dt \middle| \mathscr{F}_\tau\right] - e^{-r\tau} \leq 0,$$

with equality when aggregate investment increases. This is the profit of a nonatomic firm entering at τ, cf. Baldursson and Karatzas (1997).

5. Explicit solutions

Specify inverse demand to be of constant elasticity with a multiplicative shock.

$$P(x, q) = x \cdot q^{-\frac{1}{\alpha}},$$

with $\alpha > 0$. Further, let $X_t = e^{Y_t}$ be an exponential Lévy process without negative jumps. Then equilibrium investment tracks the shock process.

Proposition 4. *If $\alpha > \frac{1}{n}$, the unique open loop investment equilibrium is*

$$Q_t^{*_i} = \sup_{0\leq u<t} \frac{1}{n} \kappa^\alpha X_u^\alpha \quad (i = 1 \dots n)$$

with constant parameter κ.

Investment occurs whenever X sets a new record, because the shock has independent increments and positively influences profits.

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