Optimal Stopping under Ambiguity in Financial Markets

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1. Introduction and Motivation

In Finance, we face optimal stopping problems when pricing, hedging and looking for exercising rules for *American options*. An American option gives the owner the right to purchase or sell, respectively, a share of stock (the underlying) at a fixed price K (the so–called strike price) at any time during its life. In the first case it is named a call option, in the second a put option.

The holder of an American option aims to maximize her expected gain from exercising the option at an optimal time. In complete markets, the value of an American option is the value of the optimal stopping problem under the (unique) equivalent martingale measure. It can be determined by

$$\max_{\tau \le T} \mathbb{E}^{P^*}(S_\tau)$$

where P^* denotes the (risk–neutral) martingale measure, τ a stopping time, and $S = (S_t)$ the payoff process from exercising the option. (T denotes the time at which the option expires.)

When markets are incomplete there is more than one equivalent martingale measure. Thus, the market participants may face many distributions of payoffs. Also, uncertainty occurs when the information is too imprecise. In Finance, market participants usually encounter this situation: Due to the structure of financial markets, complete knowledge of all market data is rarely possible. In addition, taking into account the possibility of varying market data over time we must accept that “real world” models include ambiguity, for instance, uncertain asset price dynamics. To incorporate this uncertainty we use *multiple prior* models.

The optimal stopping problem (o.s.p.) being defined in (1) does not only serve as an evaluation method for American options. It also can be used in the context of risk measurement and robustness. In the sense of model robustness, by solving (1) one obtains exercise strategies which also perform well when model data change slightly.

In order to accommodate all cases above we consider the o.s.p. under ambiguity aversion. The famous Ellsberg paradox also justifies this point of view. It gives experimental evidence for the existence of ambiguity aversion of decision makers.

2. Optimal Stopping with Multiple Priors

We analyze optimal stopping problems under ambiguity aversion. Formally, on a given probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ the o.s.p. for an ambiguity averse decision maker is defined as

$$\operatorname*{ess\,sup}_{t \le \tau \le T} \operatorname*{ess\,inf}_{Q \in \mathcal{P}} \mathbb{E}^Q(S_\tau | \mathcal{F}_t) \qquad (1)$$

where S_τ is the payoff at stopping time $\tau \in [0, T]$. The agent aims to determine τ optimally. $\mathcal{P}$ denotes a *set of priors* (probability measures) which models the uncertainty of the agent. Due to her ambiguity aversion she considers the *worst–case scenario* and tries to stop the payoff process optimally in this scenario.

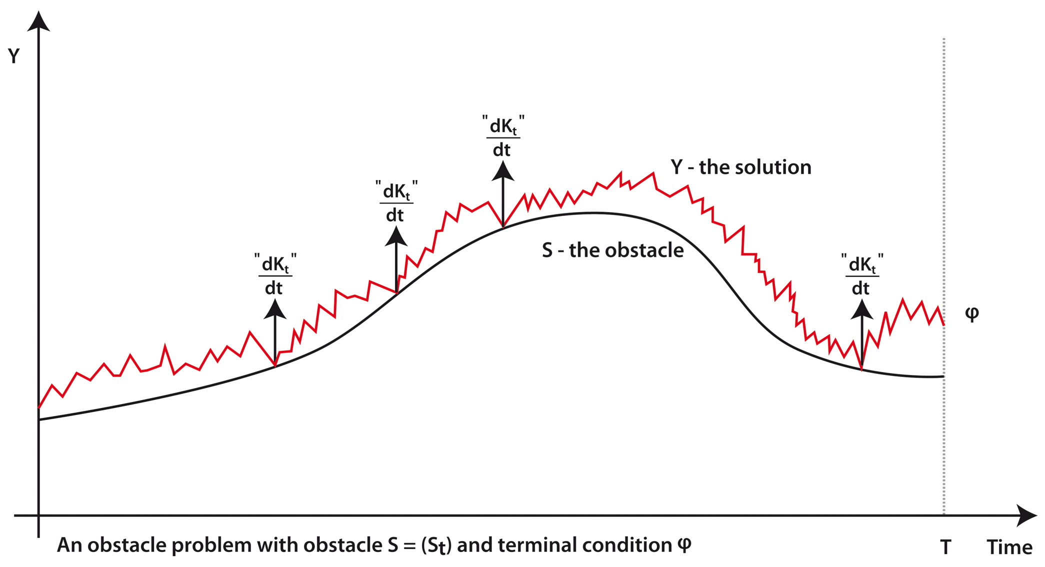
Under appropriate assumptions on the set of priors, see [3], in *discrete time* the problem can be solved by a generalized version of *backward induction*. It is defined for a payoff process $(S_t)_{t=0,\dots,T}$ with respect to $\mathcal{P}$ recursively by $U_T := S_T$ and

$$U_t := \max\left\{S_t, \operatorname*{ess\,inf}_{P \in Q} \mathbb{E}^P[U_{t+1}|\mathcal{F}_t]\right\}$$

for $t = 0, \dots, T-1$. An optimal stopping rule is given by

$$\tau^* := \inf\{t \ge 0 | U_t = S_t\}$$

see [3]. In *continuous time* reflected backward stochastic differential equation (*RBSDE*) replaces backward induction as a methodology for solving (1), see [2], [1]. Roughly speaking, it is the solution of a probabilistic obstacle problem. A solution $Y = (Y_t)$ for a given obstacle $S = (S_t)$ and a terminal condition $Y_T = \varphi$ is illustrated in the figure:



Within the framework of option pricing the obstacle S represents the payoff process of the option. Whenever $Y_t = S_t$ the solution is “pushed upwards” such that $Y \ge S \; dt \otimes P$ a.e. remains satisfied. These particular points in time specify optimal stopping times.

3. The Setting

Here we restrict the analysis of problem (1) for American options on the continuous time case. The first question is the construction of the set of multiple priors $\mathcal{P}$. For several reasons we focus on κ*–ambiguity*, see [1]. Its construction relies heavily on Girsanov’s theorem.

Let $W = (W_t)$ be a Brownian motion on $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)$ be the canonical filtration generated by W. For $\kappa > 0$ we define

$$\Theta := \{(\theta_t) | \text{ progr. mb. w.r.t. } (\mathcal{F}_t), \; \theta_t \in [-\kappa, \kappa] \; \forall t\}.$$

Then for any $\theta \in \Theta$ a new probability measure $Q^\theta \sim P$ is defined by the following rule: For any $A \in \mathcal{F}$

$$Q^\theta(A) := \mathbb{E}^P\left(1_A z_T^\theta\right) \qquad (2)$$

where $z^\theta = (z_t^\theta)$ denotes the Girsanov kernel which can be identified as the unique solution of the SDE

$$dz_t^\theta = -\theta_t z_t^\theta dW_t, \quad z_0^\theta = 1.$$

Then the set of priors is defined as

$$\mathcal{P} := \{Q^\theta | \theta \in \Theta \text{ and } Q^\theta \text{ is defined by (2)}\}.$$

All options we focus on are assumed to be written on a risky stock X evolving according to

$$dX_t = rX_t dt + \sigma X_t dW_t, \quad X_0 = x,$$

w.r.t. P, whereas r is a fixed interest rate. These dynamics change to

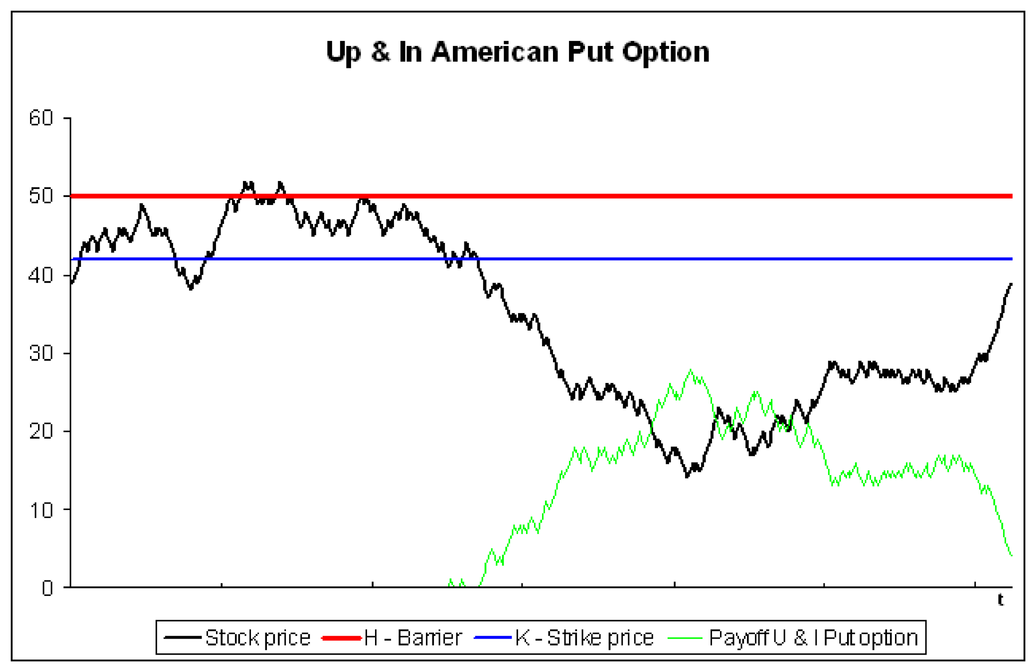
$$dX_t = rX_t dt - \sigma X_t \theta_t dt + \sigma X_t dW_t^{Q^\theta}, \quad X_0 = x,$$

when it is considered under $Q^\theta \in \mathcal{P}$. Here $(W_t^{Q^\theta}) := (W_t + \int_0^t \theta_s ds)$ is a Brownian motion w.r.t. Q^θ. This establishes the important identification that κ–ambiguity just leads to varying drifts for the stock.

By using the theory of RBSDEs we show that the ambiguity averse agent uses $Q^{-\kappa}$ for the evaluation of an American put option, and Q^κ for an American call option, respectively. Hence in the put case, the stock is assumed to exhibit the highest possible drift among all scenarios of $\mathcal{P}$, and the lowest one in the case of a call, respectively.

4. American Barrier Option

In the examples above the option’s payoff featured the same monotonicity in the underlying stock at all times. Without this kind of monotonicity of the option’s payoff the solution of the o.s.p. under κ–ambiguity is much more involved. As an illustration we analyze this in the case of an American *up–and–in put* option. This option is characterized by the necessity that the stock price needs to break through a barrier before the holder is allowed to exercise the option. The option’s payoff from exercising for a given stock price history is illustrated in the figure:



Formally, the payoff is defined as

$$S_t = \max(K - X_t, 0)\, 1_{\{t \ge \tau_H\}}$$

with the *knock-in* time $\tau_H := \inf\{t \ge 0 | X_t \ge H\}$ for a barrier level H and strike price K.

It turns out that the ambiguity averse agent evaluates the option under a prior which leads to the lowest drift for the underlying stock before knock–in, and the highest one afterwards. Hence in this setting, the agent computes her optimal strategy in (1) according to a prior under which the returns of the stock are correlated.

5. Conclusion and Further Results

On the one hand introduction of multiple priors leads to more complicated evaluation. On the other it results in dynamical model adjustments. With these the agent takes into account changing beliefs caused by realized events.

Similar results can be obtained in the situation of ambiguous interest rates. Here, the stock price dynamics evolve according to

$$dX_t = \mu X_t dt - \sigma X_t \psi_t(r) dt + \sigma X_t dW_t^{Q^{\psi(r)}}$$

and the agent faces uncertainty about the interest rate r which in turn leads to uncertainty about the *market price of risk* here denoted by $\psi = (\psi_t)$ depending on r.

References

[1] Chen, Z. and Epstein, L., *Ambiguity, Risk and Asset Returns in Continuous Time*, Econometrica, Vol. 70, 2002.

[2] El Karoui, N., Peng, S. et al., *Reflected Solutions of Backward SDE’s, and Related Obstacle Problems for PDE’s*, The Annals of Probability, Vol. 25, 1997.

[3] Riedel, F., *Optimal Stopping with Multiple Priors*, Econometrica, Vol. 77, 2009.