Product and Quality Innovations: An Optimal Control Approach

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EBIM — Economic Behavior and Interaction Models

1. Motivation

One of the basic sources of economic growth is technological progress, as it is argued by economic growth theory. Technological progress happens due to innovative activity of economic agents. That’s why modeling innovations is one of the key areas of modern economics. Starting from 1960s there been a lot of attempts of such modeling and incorporation of innovations to macroeconomic models. All of them proved to be unsuccessful till recently.

At the beginning of 1990’s two new approaches to innovations emerged, namely, Romer’s (1990) model of expanding variety of products and Aghion & Howitt’s (1993) model of quality ladders. Each of them addressed different aspects of innovative activity, but unlike of previous models, they endorsed technological change. It is argued, that both these models are complementary in nature, describing two aspects of the same process, which are going on simultaneously. At the same time, there is no a unified model, which would take into account both these aspects till today. This model tries to answer two questions, namely:

- How expansion of goods variety influences quality innovations to already existing and new products?
- What is the role of structural characteristics of these products by themselves in innovative process?

2. Model

We suggest the simple model, which try to combine product and quality innovations in one dynamical system. Range of products to be invented is assumed to be a continuum space of potential products. For sake of simplicity and tractability we omit any uncertainty at this stage. First we account for ‘planned’ innovations, where one agent is simultaneous investing in expansion of the variety of products available and in the increase of quality of every product already invented. Social planner is maximizing an objective functional:

\[ J = \int_{0}^{T} q(t,T)dt + \int_{0}^{T} \left( \frac{1}{2} n(t)^{2} + \gamma t \right) dt \rightarrow \max \]  

subject to dynamic constraints on product quality and goods innovations:

\[ n(t) = \alpha u(t), \]
\[ q(t) = n(t) \gamma(t) \beta i(t) - \beta i(q(t)), \forall i \in \{0, \ldots, N\}, \forall t \in [0, T], \beta > 0, \gamma \in \mathbb{R} \]
\[ s(t, n(t)) = \begin{cases} 1, & n(t) \geq i; \\ 0, & n(t) < i. \end{cases} \]

Where \( q(t) \) are investments to the \( i \) product quality, \( u(t) \) are investment to invention of new products.

There is a continuum of potential products and one have uncommently many dynamic constraints for quality growth of every product \( i \).

Planer is maximizing:

\[ J_{\text{planner}} = \int_{0}^{T} q(t,T)dt - \frac{1}{2} \int_{0}^{T} u(t)^{2}dt \rightarrow \max \]  

subject to one constraint on the products’ quantity \( n(t) \) from above.

3. General Results

First we establish the structure of the state space and control set. For competitive problem one have just \( N + 1 \)-dimensional states and equal number of control problems. Note, that \( N \) is uncountable and thus is interpreted as range of products rather then their actual number. For planned innovations model one have an infinite-dimensional state space and control set.

**Theorem 1 (State Space)**

In planned innovations model state space is an \( L^{2} \times \Omega \) space.

Further on we prove existence of optimal controls for the model. This can be done due to the fact, that the control space is a compact space.

**Theorem 2 (Existence Theorem)**

In both specified models there exist sets \( \{w_{1}(t), q_{1}(t)\} \) and, accordingly, \( \{w_{2}(t), q_{2}(t)\} \), \( \forall i \in \mathcal{I} \) such that they maximize (1) and (2), (3) respectively.

These general statements allows to obtain solution to the model via Maximum Principle.

4. Solutions

One can solve specified above model in parametric form, when functions \( \gamma(t) \) and \( \beta(t) \) are not given. Quality investments are independent of the market structure starting from time when the product is invented. Time of invention, however, is determined from \( n(t) \) dynamics:

\[ q(t, n) = \frac{q(t, 0, n(t) < i; \forall i \in \mathbb{R}, \forall t \in [0, T], \beta > 0, \gamma \in \mathbb{R}}{\gamma(t) s(t, n(t)) = \begin{cases} 1, & n(t) \geq i; \\ 0, & n(t) < i. \end{cases} \]

Where \( \delta_{i,j} \) are constants depending on terminal time and parameters. New products emerge continuously and once invented, investments to their quality start. Efficiency of investments is defined by \( \gamma(t) \) function. Speed of decline of quality in the absence of investment is defined through \( \beta(t) \) function.

Analysis of dynamics of quality reveals, that every product’s quality dynamics is of saddle type with some maximal attainable quality level. This maximal level is different across products and actual form of 2-dimensional distribution of stable qualities depends on the specification of \( \gamma(t) \) function. Speed, at which new products emerge on the market is defined through \( n(t) \) dynamics. In parametric form equation for \( n(t) \) has the solution which is a function of time and \( \gamma(t) \) function:

\[ n(t) = \int_{0}^{T} \gamma(n(t)) * e^{\beta(t)n(t)T} \]  \( t \) turns out, that dynamics of product innovations depend not from all previously invented products parameters, but only from the recently invented one. Thus one have a notion of innovations’ frontier, defined through parameters of frontier products, which are constantly changing, at every point in time.

To obtain an explicit solution for \( n(t) \) one have to specify two functions. The simplest specification is

\[ \gamma(t) = V_n(t) = (N - t) \times \gamma. \]

In these form one have second order linear non-autonomous ODE for \( n(t) \) dynamics in both versions of the model, which does have a closed form solution in elementary functions. For planned innovation case it is:

\[ n(t) = r \times n(t) + C(t) \times (N - n(t)) \]  \( C(t) \) is the time-varying parameter. This is the linear ODE, which always has a solution. We integrate it numerically and obtain qualitative behavior in 3-dimensional phase-space:

Dynamics of products’ variety and quality can be reconstructed in 3 dimensions:

\[ \text{Conjecture 1 (Wave Effect)} \]

Range of products available is changing as a wave-generating function. Moreover, quality dynamics of all products are interconnected as different waves in space of goods and \( q(t, n) \) is a distribution function of these waves.

Some more technical work is needed to demonstrate it rigorously.

References