Simultaneous Search and Network Efficiency*

Pieter A. Gautier† and Christian L. Holzner‡

Abstract

When workers send applications to vacancies they create a bipartite network. Coordination frictions arise if workers and firms only observe their own links. We show that those frictions and the wage mechanism are in general not independent. Only wage mechanisms that allow for ex post competition generate the maximum matching on a realized network. We show that random search with ex post competition in wages leads to the maximum number of matches and is socially efficient in terms of vacancy creation, worker participation and the number of applications send out, if workers and not firms have the power to make offers.

Keywords: Efficiency, network clearing, random network formation, simultaneous search.

JEL-Classifications: D83, D85, E24, J64

*The authors gratefully acknowledge the hospitality of CESifo and Christian Holzner acknowledges the financial support by the German Research Foundation, grant Ho 4537/1-1. The authors thank Philipp Kircher, René van den Brink seminar participants at CES-ifo, the University of Edinburgh, the University of Essex, the University of Konstanz, the University of Mainz, the Norwegian School of Management, the 2011 Tinbergen conference, the University of Amsterdam and VU Amsterdam, (in particular). We also thank Xiaoming Cai for his excellent assistance in programming the decomposition algorithm.

†VU University Amsterdam, Tinbergen Institute, CEPR, IZA, email: p.a.gautier@vu.nl

‡University of Munich, Ifo Institute for Economic research and CESifo, email: holzner@ifo.de.

1 Introduction

When workers apply to one or more jobs, a network arises where each application establishes a link between a worker and a firm. In such a decentralized environment there are two coordination frictions, (i) workers do not know where other workers apply to and (ii) firms do not know which workers are considered by other firms. We can think of the first coordination friction as referring to random network formation, while the second coordination friction affects network clearing (the number of matches on a given network). Treating the job search process as a matching on a bipartite graph (network) gives new insights into one of the key questions in the labor-search literature namely, under which conditions is the decentralized market outcome constrained efficient? With constrained efficiency we mean that the market outcome is identical to the outcome of a hypothetical social planner who maximizes social welfare given the fundamental frictions (i) and (ii).

The main contribution of our paper is that it shows how the wage mechanism determines frictions through network formation and clearing.[1] We find that efficient network formation is ensured if identical vacancies have the same application arrival rate (this implies no ex ante wage dispersion) and that efficient network clearing requires ex post competition between firms that consider the same candidate. We present the explicit matching function based on the results of Frieze and Melsted (2012), which reduces to the well known urn-ball matching function in case only one application is send out. Given this matching function we show that vacancy creation, worker participation and the search intensity is efficient, if the Mortensen rule (1982) holds, i.e., if workers are awarded the worker-maximizing point in the core.

Wage mechanisms that allow for ex post competition generate the maximum number

[1] Coles and Eeckhout (2003) and Eeckhout and Kircher (2010) show that the number of matches in a model with identical workers is independent of the posted wage mechanism. We show that this no longer holds if workers send multiple applications. When workers apply to only one job, only the first coordination friction occurs, since all firms that receive at least one application can be sure that their selected candidate has no competing offer from another firm, see Burdett, Shi and Wright (2001). In the random search models of Diamond (1982), Mortensen (1982) and Pissarides (2000) the wage determination process and the matching process are fully independent. In Moen's (2000) competitive search model, workers can sort in submarkets which are characterized by different wage and market tightness pairs. Within each submarket, given market tightness, the number of matches does not depend on wages.

of possible matches on a realized network and are therefore socially efficient. This happens because firms can respond to a particular realization of the network by increasing their posted wages. Specifically, firms that have n candidates who are collectively linked to more than n firms will bid more aggressively than firms with n candidates who are collectively linked to less than n firms. Finally, we show how in a decentralized economy, workers and firms can reach the maximum number of matches through offers and counter offers. The mechanism only requires the agents to know their own links and not the entire network.

Our paper is the first one that analyzes how standard decentralized wage mechanisms affect network formation and network clearing in a decentralized search model with complete recall where workers only know to which firms they applied and firms only know which workers applied to them. The only other paper that we are aware of that considers a search model with multilateral negotiations where workers and firms do not know the entire network is Elliot (2011). He focuses on the efficiency of entry and search intensity and allows for heterogeneity. In his wage mechanism, the bargaining power is assumed to be independent of the type of subgraph an agent is in, while we show that the type of subgraph determines the agents' payoff.[2] Manea (2011) considers a framework where agents who are connected in a network are randomly selected to bargain. During the bargaining game they are not able to contact other connected agents. His random selection setting implies that a firm with many candidates has a stronger bargaining position, because it is more likely to be selected. In our model it is not the number of candidates that matters but whether a firm is located in a subgraph with more firms than workers.

Part of the network literature has analyzed different pricing mechanisms and has studied whether these price mechanisms lead to an efficient matching of sellers and buyers. Kranton and Minehart (2001) show for example that a public ascending price auction ensures efficient network clearing. Corominas-Bosch (2004) shows for identical sellers and buyers that an alternating-offers game where all sellers (or buyers) of a subgraph simultaneously announce prices, leads to a maximum matching. This literature, however, assumes that once a network has been formed, all agents know the complete network

[2]Following Corominas-Bosch (2001) each graph can be decomposed into worker subgraphs with an excess number of workers, firm subgraphs with an excess number of firms and even subgraphs with an equal number of workers and firms.

(or the entire subgraph of the network they are in).[3] This knowledge allows sellers and buyers to determine their exact outside option (trading partners and trading prices). We show that ex post competition achieves the maximum matching, even if agents do not know the network structure. Another part of the network literature uses the set-valued approach, i.e., it either starts with a set of competitive price vectors and shows that the resulting matches are pairwise stable and maximize aggregate welfare (see Kranton and Minehart, 2000), or it starts by assuming that pairwise stable matches must arise and then analyses wage formation (see Elliott, 2011). Those papers do not layout the game that leads to a competitive price vector or a pairwise stable matching like we do. Moreover, pairwise stable matchings are not necessarily maximum matchings (i.e., Kircher, 2009) but a maximum matching is always stable since an improvement of one agent must make another agent worse off. Finally, there is a growing number of papers that combine insights from search and network theory.[4] Those papers focus mainly on how social networks of workers can pass information of the location of jobs on to each other, which is very different from the bipartite network (between workers and firms) framework that we consider here.

We also

Finally, we look at entry and search intensity and generalize the Kim and Kircher (2012) result that efficiency requires that workers receive the worker-optimal point in the core and that this requires that workers receive the full surplus in even subgraphs (that have an equal number of workers and vacancies). This solution can be decentralized by reversing the role of workers and firms. The fact that there exists a decentralized wage mechanism that is efficient in all dimensions is surprising because it requires two externalities to exactly offset each other. The first externality is a business-stealing externality caused by the fact that a new vacancy does not internalizes that it reduces the hiring probability of other firms by making it more likely that other firms end up in a firm graph (firms are collectively linked to less workers than the number of firms in the subgraph). The positive externality is caused by the fact that a firm that ends up in an even subgraph

[3] Galeotti et al. (2010) analyse network games with limited information. However, they only consider one type of agents, i.e., they do not consider vacancies and workers or sellers and buyers in a bipartite network.

[4] Example include, Boorman (1975), Calvó-Armengol and Jackson (2004), Calvó-Armengol and Zenou (2004), Fontaine (2004).

adds social value but receives nothing if the worker gets the full surplus in even subgraphs.

The paper is organized as follows. We start with a description of the model setup in section 2. In section 3 we illustrates with a 3-by-3 example our main point that random search with ex post competition in wages will maximize the expected number of matches. In section 4 we take the realized network that forms through the random application process as given and solve the assignment game with ex post competition in wages. We also show that ex post competition is socially desirable, since it leads to a maximum matching. Section 5 solves for the matching function that result form the random application process. Within this section we also proof that random search is indeed socially efficient. Section 6 provides the conditions under which the search intensity (number of applications) chosen by workers and the entry of vacancies and workers are socially efficient. Finally section 7 concludes.

2 Framework

We consider v identical firms with one vacancy each and u identical risk neutral unemployed workers, who send $a \leq v$ applications to different firms. Search is random, i.e., workers send each application with probability $1/v$ to any specific vacancy. Workers have a reservation wage of 0 and a matched firm-worker pair produces 1. We take the number of applications that workers send out and the market tightness as given. The main reason for this is that the conditions for efficient entry and the number of applications are well known and have been studied before.[5] This allows us to focus on the efficiency of random network formation and network clearing. In section 7 we will make some remarks on the efficiency of firm entry and search intensity.

The realized network that is formed by the random application process is unknown to workers and firms. The following time path of the assignment game also describes the action and information sets of workers and firms:

[5] Gautier and Moraga-Gonzalez (2005) and Albrecht et al. (2006) find without recall, that workers send too many applications (due to rent seeking and congestion externalities) and that entry is excessive, because firms have too much market power. Kircher (2009) shows that with directed search, wage commitment and full recall, entry and search intensity are socially efficient. Elliot (2011) finds that firm entry is never excessive but can be too small and that workers send too many applications.

1. Each firm selects one worker (if present) and offers that worker a wage $w \geq 0$. Wage offers are discrete $w \in \{\Delta, 2\Delta, ..., 1-\Delta, 1\}$, where Δ is a small but discrete amount, i.e., a cent.

2. A worker with one or more offers can keep at most one offer (which is observed and verifiable by all linked firms) and must reject all others. The worker and the only not rejected firm are labeled to be *engaged*.

3. The firms that are rejected select a worker (possibly the same worker) and offer that worker a wage $w \geq 0$ given the wage offers from other firms that are kept by their applicants.

4. The engaged firms can make a counter offer.

5. A worker with one or more offers can keep at most one offer (observed by all linked firms) and must reject all others.

6. Return to stage 3)... until the final round T (sufficiently large).

We assume the following tie breaking rule for workers. Workers keep the offer of the engaged firm if it offers the same wage as the highest offer made by any other firm. If the worker was not engaged and two or more outside firms offer the same highest wage, the worker randomly picks one of them and rejects the others.

The network clearing game assumes that firms have all the bargaining power and can make take-it-or-leave-it wage offers. An interesting alternative is the case where workers have all the bargaining power. In that case, the network clearing game is the same as above except that we have to exchange the roles of firms and workers (workers now make a take-it-or leave-it offer, $w \leq 1$).[6]

When investigating the efficiency property of random search with ex post competition in wages we have to constrain the social planner, since an unconstrained social planner will trivially assign each unemployed worker to a vacancy such that the number of matches

[6]Kim and Kircher (2012) show in a directed search setting with $a = 1$ that from a welfare point of view this is more desirable.

equals the short side of the market. During the random network formation game we constrain the social planner to choose the set of vacancy subgroups C (where each subgroup c is defined by a certain color), the measure of vacancies v_c within each subgroup c and the probability p_c that a worker sends a particular application to subgroup $c \in C$. Thus, the social planner can choose between random search (application probabilities p_c are the same across subgroups) and directed search (application probabilities p_c differ across subgroups). Once the network has been formed the social planner can decide on the wage mechanism used to solve the assignment game (network clearing game).

3 3-by-3 example

This section illustrates our main points that (i) random search compared to directed search with ex ante wage dispersion leads to efficient network formation and that (ii) a wage mechanism with ex post competition in wages generates a maximum matching compared to a wage mechanism with ex ante wage commitment. In this simple environment 3 workers send 2 applications to 3 firms without knowing where other workers apply to.[7]

First, consider network clearing. Efficient network clearing requires that the number of matches is equal to 3, if all three vacancies are collectively linked to all three workers, and that the number of matches is equal to 2, if only two vacancies are collectively linked to all three workers. Note that these are the only two possible outcomes, since no worker sends both applications to the same firm. Network clearing is in general not efficient, if firms commit to their posted wages. To see this, consider the graph in Figure 1, which pictures a particular realization of the case where each worker sends one application to the high-wage firm and one to one of the two low-wage firms (thick lines).

The number of matches (dashed lines) now depends on which worker is chosen by the high-wage firm. If the high-wage firm offers the job to one of the workers who are linked to the low-wage firm with two applicants, i.e., to worker 2 or 3 in Figure 1, the number of matches is equal to the maximum number of matches (3). If the high-wage firm offers the job to the worker linked to the low-wage firm with only one applicant, i.e. to worker 1

[7] If workers send 1 application or 3 applications, the number of matches generated is independent of the wage mechanism used.

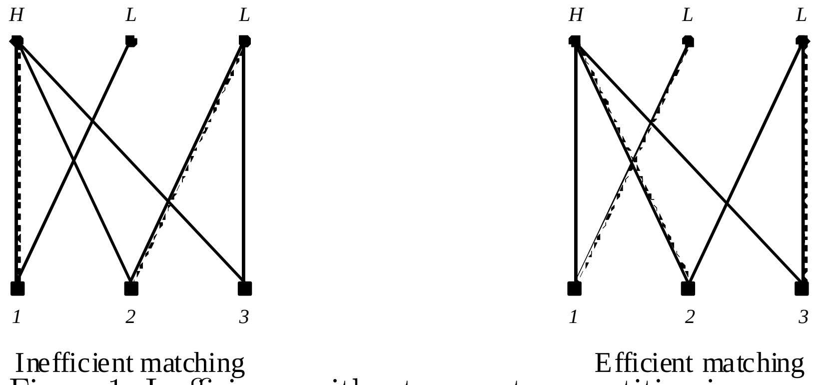

Figure 1: Inefficiency without ex post competition in wages

in Figure 1, there are only two matches, since the low-wage firm with only one applicant will remain unmatched. If this firm *could* ex post increase its initial offer it would bid the high wage plus epsilon and hire worker 1 while the high-wage firm would hire one of its other candidates. Following the assignment game laid out in section 2 it is easy to show that in this example, allowing for ex post competition always leads to the maximum number of matches.

Now, let us look at random network formation and assume that network clearing generates the maximum number of matches. Denote by $\xi_i = ap_i = 2p_i$ the probability that a worker sends at least one of her two applications to vacancy i. Under the assumption that network clearing is efficient, the expected number of matches is,

$$M = \sum_{i=1}^{3} \left(1 - (1-\xi_i)^3\right), \text{ with } \sum_{i=1}^{3} \xi_i = 2,$$

where $(1-\xi_i)^3$ equals the probability that vacancy i does not get any application. Since the function $\left(1 - (1-\xi_i)^3\right)$ is concave in ξ_i, Jensen's inequality implies that the number of matches is maximized, if all vacancies have the same probability to receive an application, i.e., if $\xi_i = 2/3$ or $p_i = 1/3$. Thus, random search leads to the maximum number of matches, $M = 26/9 \approx 2.889$, while directed search with $p_i \neq 1/3$ will leads to less matches.

This illustrates that the wage mechanism and the matching process are not independent. Different search environments generate different distributions of networks and whether the wage mechanism allows for ex post competition or not affects the number of matches for a given network. A final important point is that both wage mechanisms, i.e., ex post competition in wages and wage commitment, generate a stable matching.

4 Network clearing

In section 4.1 we solve the assignment game using a decentralized wage mechanism with ex post competition. In section 4.2 we show that the wage mechanism with ex post competition leads to a maximum matching while ex ante wage commitment generally fails to achieve the maximum matching on the realized network.

4.1 Assignment game

The time structure of the assignment game has already been laid out in section 2. Before we are able to solve the assignment game, we need to introduce some concepts of graph theory that allow us to determine the optimal strategy and describe the equilibrium.

4.1.1 Network decomposition

In order to determine the optimal strategies for workers and firms we use the properties of the Decomposition Theorem by Corominas-Bosch (2004) (for details see Appendix B), which – in terms of our terminology – decomposes a realized network into firm-, worker- and even subgraphs. A firm subgraph contains more firms than workers. A worker subgraph contains more workers than firms. In even subgraphs, the number of workers equals the number of firms (see Figure 2).The decomposition algorithm first looks for firm subgraphs and separates all of them from the network. Then it identifies worker subgraphs and removes all of them from the network. The remaining subgraphs are even subgraphs. The decomposition is not unique but the Decomposition Theorem states that any firm and any worker will always belong to the same type of subgraph, a property important to guarantee that the different possible decompositions are payoff equivalent.

Figure 2 illustrates the Decomposition Theorem. The algorithm starts with the first firm and identifies a set of firms as firm subgraph if it has less neighbors (more precisely, if it is jointly linked to less neighbors, i.e., $|F| > |N(F)|$). In order to ensure that the maximum matching is found, the algorithm has to start with $|F| = 1$. The number $|F|$

increases by one once all firm combinations with $|F|$ have been considered (Hall's Theorem, 1935). The first subgraph in Figure 2 is the unmatched firm G. The firm subgraph G_1^f is removed before the algorithm continues. Since there are no firm subgraphs with $|F| = 2$, the next firm subgraph has three firms, i.e., $|F| = 3$, The three firms A, B and C in this subgraph are collectively linked to workers 1 and 2, i.e., $N(\{A, B, C\}) = \{1, 2\}$ and $|N(\{A, B, C\})| = 2$. Once the firm-subgraph G_2^f is removed, it is easy to verify that the remaining sets of firms are collectively linked to more neighbors, i.e., $|F| \leq |N(F)|$. Hence, there are no further firm subgraphs. The algorithm continues by looking for worker subgraphs in the same way as it looked for firm subgraphs. At $|W| = 4$, the algorithm identifies a worker subgraph with $N(\{3, 4, 5, 6\}) = \{D, E, F\}$ and $|N(\{3, 4, 5, 6\})| = 3$. Once the worker subgraph G_1^w is removed, and no further worker subgraphs are found the algorithm stops by identifying all remaining subgraphs as even subgraphs, i.e., in Figure 2 the remaining subgraph G_1^e is an even subgraph with $N(\{7, 8\}) = \{H, I\}$ and $|N(\{7, 8\})| = 2 = |\{H, I\}|$.

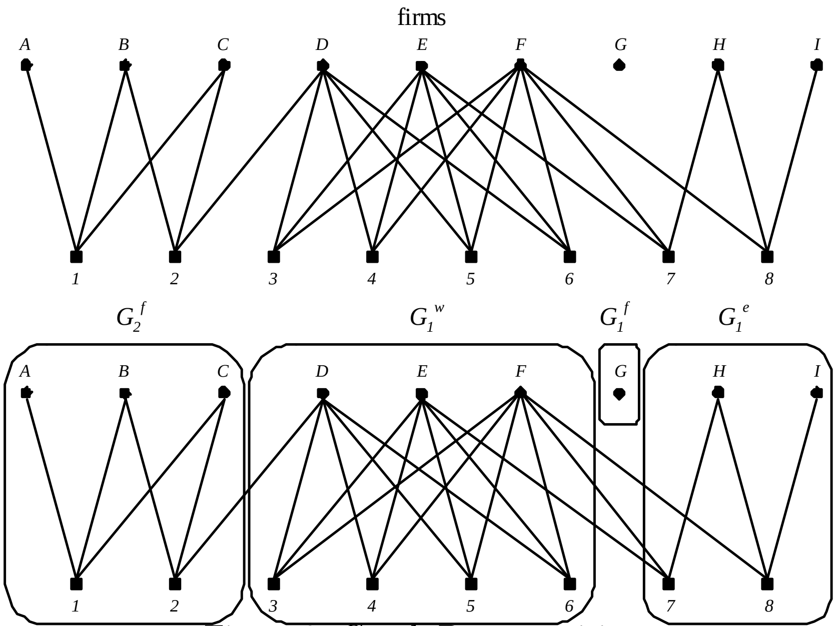

Figure 2: Graph-Decomposition

Figure 2 illustrates an important property of the resulting subgraphs (compare Corominas-Bosch, 2004, p. 51). The long side of a subgraph has only links to the short side of the respective subgraph, i.e., the workers in a worker-subgraph, G_i^w (firms in a firm-subgraph,

G_i^f) are only linked to firms (workers) in a worker (firm) subgraph. Workers in an even subgraph, G_i^e, are only linked to firms in worker- or even subgraphs and the firms in even subgraphs can only be linked to workers in firm- or even subgraphs.

4.1.2 Information sets and beliefs

The actions of firms (or more general, the agents who have the power to propose the wage) will depend on their belief about the subgraph they are in. Firms will update their beliefs given the number of applicants N they have and the wage offers of their applicants, i.e., the set of wage offers W^N the applicants hold from their engaged firms. Denote the belief of firm j in round t that it is in a firm subgraph given N and W^N, by $b_{j,t}\left(N, W^N\right)$. I.e., if a firm is sure to be in a firm subgraph $b_{j,t}\left(N, W^N\right) = 1$. Firms without any applicant are by definition in a firm subgraph (see G_1^f in Figure 2). Firms with at least one applicant can (initially) be in any type of subgraph.

We will show below that firms can infer from the (sub)sets of observed wage offers W^N whether they are in a firm subgraph or not. Let us therefore define the following subsets. If $k \in \{0, 1, ..., N\}$ applicants receive no offer, denote the respective subset of wage offers by Ø^k. If $l \in \{0, 1, ..., N\}$ applicants hold wage offers equal to zero, denote the respective subset by 0^l. If $m \in \{0, 1, ..., N\}$ applicants hold an offer equal to one cent, denote the respective subset by Δ^m. Finally, if $q \in \{0, 1, ..., N\}$ applicants hold an offer equal to one, denote the respective subset by 1^q. Thus, the set of wage offers is equal to $W^N = \left\{\widetilde{W}^{N-k-l-m-n}, \text{Ø}^k, 0^l, \Delta^m, 1^q\right\}$, where $\widetilde{W}^{N-k-l-m-q}$ equals the remaining subset of wage offers with wages $w \in \{2\Delta, 3\Delta, ..., 1-\Delta\}$. We show below that in equilibrium $\widetilde{W}^{N-k-l-m-n}$ is empty in the final round given that T is sufficiently large.

4.1.3 Workers' and firms' strategies

Below, we prove that the following set of strategies constitutes a perfect Bayesian Nash equilibrium to the assignment game.

Consider the following worker strategies:

A1 In the final round T, accept the best offer.

A2 In any previous round $t < T$, keep the best offer $w^h = \max\{w^1, w^2, ..., w^a\}$, and reject all other offers. Accept the best offer, if $w^h = 1$.

Engaged firms have one advantage over rejected firms. They can make a counter offer before the next round starts. This implies that they can base their actions on the set of wage offers W^N they observe and do not need to base their actions on the beliefs about the type of subgraph they are in. We therefore consider the following counter offer strategies for engaged firms:

B1 In the final round T, match any offer.

B2 In any previous round $t < T$, match any outside offer $w^h = 0$. Match the offer $w^h \geq \Delta$, if all other applicants hold an offer $\widetilde{w}^h \geq w^h - \Delta$, and don't match the offer $w^h \geq \Delta$, if at least one other applicant holds no offer or an offer $\widetilde{w}^h < w^h - \Delta$.

For rejected firms in rounds $t < T$ we consider strategies that are independent of the firm's belief $b_{j,t}\left(N, W^N\right)$ and only in the final round T we consider strategies that depend on the belief $b_{j,T}\left(N, W^N\right)$.

C1 In the final round T, the strategy is as follows:

C1a If at least one applicant holds no offer, i.e., $W^N = \left\{\widetilde{W}^{N-k-l-m-q}, \emptyset^k, 0^l, \Delta^m, 1^q\right\}$ with $k > 0$, then offer one of the k applicants $w = 0$ if $b_{j,T}\left(N, W^N\right) = 0$, else offer one applicant $w \in F(w)$ if $b_{j,T}\left(N, W^N\right) \neq 0$, where the optimal wage offer distribution $F(w)$ is characterized in Gautier and Moraga-Gonzalez (2004). (Note, that we show below that for T sufficiently large $b_{j,T}\left(N, W^N\right) = 0$ if $k > 0$.)

C1b If all applicants hold an offer, select one worker and offer him $w = 1$ (irrespective of $b_{j,T}\left(N, W^N\right)$).

C2 In any previous round $t < T$, the strategy is as follows:

C2a If at least one applicant holds no offer, i.e., $W^N = \left\{\widetilde{W}^{N-k-l-m-q}, \emptyset^k, 0^l, \Delta^m, 1^q\right\}$ with $k > 0$, offer one of the k applicants $w = 0$ irrespective of the belief $b_{j,t}\left(N, W^N\right)$.

C2b If all applicants hold an offer $w^h \geq 0$, i.e., $W^N = \left\{ \widetilde{W}^{N-k-l-m-q}, \emptyset^k, 0^l, \Delta^m, 1^q \right\}$ with $k = 0$, offer the worker with the lowest best offer $\underline{w}^h = \min W^N$ the wage $w = \underline{w}^h + \Delta$ if $\underline{w}^h < 1$ irrespective of the belief $b_{j,t}\left(N, W^N\right)$ and $w = 1$ if $\underline{w}^h = 1$.

4.1.4 Wages and beliefs

In order to show that the proposed strategies are indeed optimal it will be useful to analyze first the wages that are paid in each type of subgraph.

Lemma 1 *Workers' and firms' strategies imply that*
(i) at $t \geq u$ all firms in worker subgraphs are engaged and their engaged workers hold an offer no higher than $w = 0$,
(ii) at $t \geq 2u$ all workers and firms in even subgraphs are engaged and all workers hold an offer $w \in \{0, \Delta\}$,
(iii) at $t \geq u/\Delta$ all workers in firm subgraphs are engaged and hold an offer $w = 1$.

Proof. See Appendix A.1.

Following the strategies above firms (workers) in worker (firm) subgraphs will receive the whole surplus, since they are collectively linked to less workers (firms). Thus, workers in worker subgraphs will receive a wage offer no higher than $w = 0$ and workers in firm subgraphs will eventually (after sufficiently many rounds) be offered the marginal product, i.e., $w = 1$. Since a rejected firm offers its job in the next round to a candidate without an offer (if present), it takes at most $t = u$ rounds until all firms in a worker subgraph are engaged (because there are more workers than firms). By the same argument all workers in firm subgraphs are engaged after $t = u$ rounds. Since rejected firms in firm subgraphs only offer Δ more than the lowest offer made by their competitors $\underline{w}^h$, it takes at most $t = u/\Delta$ rounds (starting from an initial offer of $w = 0$) until all workers hold an offer $w = 1$. The wage result for even subgraphs is due to the assumption that only firms can make offers (have all the bargaining power). Since two firms might initially compete for the same worker, it can be the case that a rejected firm (with no other option) has to offer a wage $w = \Delta$ in order to ensure that it gets engaged with the worker.

Lemma 1 implies that all firms in worker and even subgraphs are engaged at wages $w \in \{0, \Delta\}$ after $t = 2u$ rounds. Thus, a firm that remains rejected after $t = 2u$ rounds although it offered a wage $w = \Delta$ can infer that it is part of a firm subgraph. This is stated in part (i) in the following Lemma. Lemma 1 also shows that after $t \geq u/\Delta$ rounds have passed, all workers in firm subgraphs will have received an offer $w^h = 1$ and have accepted it such that all other firms that are not part of a firm subgraph can infer from the absence of wage offers $w^h = 1$, i.e., from $q = 0$ at $t \geq u/\Delta$, that they are not in a firm subgraph. This is stated in part (ii) in the following Lemma.

Lemma 2 *(i) Rejected firms that observe that their applicants hold wage offers* $W^N = \{W^{N-m-n}, \Delta^m, 1^n\}$ *with* $k = l = 0$ *in round* $t \geq 2u$ *hold a belief*

$$b_{j,t}\left(N, \left\{W^{N-m-n}, \Delta^m, 1^n\right\}\right) = 1.$$

(ii) At $t = u/\Delta$ *all firms that observe* $W^N = \left\{\not{O}^k, 0^{N-k-m}, \Delta^m\right\}$ *hold a belief*

$$b_{j,u/\Delta}\left(N, \ \left\{\not{O}^k, 0^{N-k-m}, \ \Delta^m\right\}\right) = 0.$$

Proof. See Appendix A.2.

4.1.5 Equilibrium of the assignment game

From Lemma 2 we know that $T = u/\Delta$ is sufficiently high to ensure that firms can infer whether or not they are in a firm subgraph. We therefore set $T = u/\Delta$.

Proposition 1 *With* $T = u/\Delta$ *the strategy profile A1-A2, B1-B2 and C1a-C2b constitute a perfect Bayesian Nash equilibrium to the assignment game.*

Proof. See Appendix A.3

Clearly, the workers' strategy A1 to accept the best offer in round T maximizes the workers' payoff. The same it true for the engaged firms' strategy B1 of matching any outside offer in round T, since engaged firms that do not match outside offers would remain idle and earn a profit of zero. A rejected firm finds it optimal to condition its action on its belief of whether or not it has some competitions. If it believes to have no competitors, then strategy C1a, i.e., to offer $w = 0$ to a candidate without an application, is optimal. If it has a belief $b_{j,t}\left(N, W^N\right) \in (0,1)$ it is optimal to follow the action implied by the second part of strategy C1a as characterized in Gautier and Moraga-Gonzalez (2004). Finally, if all applicants hold an offer the firm will make zero profits, since any offer will be matched by the engaged firm (see B1). It is therefore equally profitable to follow strategy C1b and offer $w = 1$.

In any round $t < T$, the workers' strategy A2 to keep the best offer w^h is a dominant strategy, because the rejected firms' strategies C2a and C2b imply that keeping a lower offer can lead to a lower payoff for the worker without increasing the chances of receiving better offers in the future. For engaged firms in rounds $t < T$ strategy B2 to match offers $w^h \geq \Delta$, if all other applicants hold an offer $\widetilde{w}^h \geq w^h - \Delta$, and not to match offers $w^h \geq \Delta$, if at least one other applicant holds no offer or an offer $\widetilde{w}^h < w^h - \Delta$, is optimal. To see this note that any deviation in the case where all other applicants hold an offer $\widetilde{w}^h \geq w^h - \Delta$ implies that the currently engaged firm has to offer $\widetilde{w}^h + \Delta = w^h$ to the other candidate without the certainty to become engaged again (since other firms might also offer w^h). If one of the other applicants holds no offer, matching the offer $w^h \geq \Delta$ cannot be optimal, because the worker holding the offer $w^h \geq \Delta$ must be part of an even or firm subgraph, while the applicant that does not hold an offer can also be part of a worker subgraph, which generates in expected terms a higher profit. If the other candidates hold an offer $\widetilde{w}^h < w^h - \Delta$, the currently engaged worker must be part of a firm subgraph, since $\widetilde{w}^h \geq 0$ implies $w^h > \Delta$. Thus, also the competing firm that offered $w^h > \Delta$ must be part of a firm subgraph. Since the competing firm in the firm subgraph will eventually pay a wage $w = 1$, it is optimal for the engaged firm not to compete, i.e., not to match w^h, but to offer the job to another applicant at the wage $\widetilde{w}^h + \Delta < w^h$. Finally, the strategies C2a and C2b of rejected firms, i.e., to pick (one of) the applicant(s) with the lowest offer and to offer this applicant the job at the lowest possible wage, are also optimal, since any deviation can potentially lead to lower profits.

Note, that the option for engaged firms to make counter offers is crucial to rule out strategic behavior of rejected firms in order to manipulate the belief of other engaged firms. Suppose for example there are 3 firms (A,B,C) in a firm subgraph with 2 workers (1,2) who applied to all three firms. Suppose firm A's wage offer $w = \Delta$ has been rejected. Then, it infers according to Lemma 2 that it is in a firm subgraph. Firms B and C will continue to believe they are in an even subgraph and offer $w = \Delta$ (or $w = 0$) to workers 1 and 2 as long as their workers do not show better offers. Why is it then not in the interest of firm A to make no offer till $T - 1$ and then offer 2Δ in round T? This is not profitable because in that case firms B or C will match this offer. Since firm A will not be able to engage with one of the workers in the last round, it may as well immediately make an offer 2Δ in the next round and thereby signal firms B and C that they are in a firm subgraph.

4.2 Maximum matching

Given that the wage mechanism with ex post competition solves the assignment game, let us now investigate the efficiency property of ex post competition.

4.2.1 Berge's Theorem

In order to be able to use proof that ex post competition leads to a maximum matching let us briefly describe some basic concepts of graph theory that we are going to use. When workers apply to jobs, each of their applications is a link (or edge) in a bipartite network (or graph). The networks (or graphs) from the random application process are simple (workers do not send multiple applications to the same firm), undirected (if worker i is linked to firm j, then firm j is linked to worker i) and bipartite ($G = \langle u \cup v, L \rangle$ consists of a set of nodes formed by two different kind of agents, i.e., by workers and vacancies, and a set of links L where each link connects a worker to a firm, so workers are not linked to other workers and firms are not linked to other firms).

Definition 1 *A matching M in a graph G is a set of links such that every node of G is in at most one link of M.*

Central to our result that ex post competition leads to a maximum matching is the following theorem by Berge.

Berge's Theorem (1957):
A matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path.

In our bipartite graph environment an M-augmenting path is defined as a path where

1. worker-firm links that are part of the matching M alternate with worker-firm links that are not part of the matching M (definition of an M-alternating path) and

2. neither the origin (firm or worker) nor the terminus (worker or firm) of the path is part of the matching M.

Figure 3 depicts an M-alternating path and an M-augmenting path in a particular network. The dots represent vacancies and the squares unemployed workers. The solid lines represent applications ($a = 2$) and the dashed lines represent matched worker-firm pairs. The M-alternating path in the first panel ($A - 1 - B - 2 - C - 4$) starts with the matched vacancy A and ends at the matched worker 4. The M-augmenting path ($A - 1 - B - 2 - C - 4$) in the second panel of Figure 3 starts with an unmatched vacancy A and ends with an unmatched worker 4.

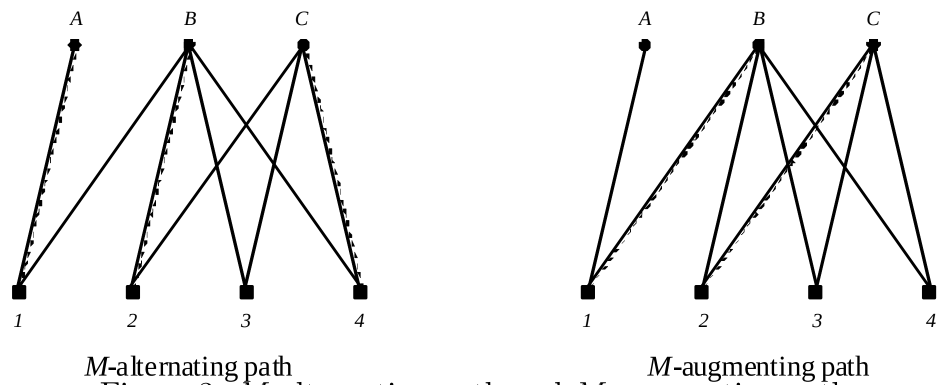

Figure 3: M-alternating path and M-augmenting path

Berge's Theorem, translated to our setting, implies that a maximum matching in a graph is only guaranteed, if an unmatched firm is not linked to an unmatched worker via an M-augmenting path. The reason that a matching is not optimal, if an M-augmenting path exists, is that one could create one more match by switching the links from not being in the matching to being in the matching and visa versa. Then, the unmatched firm at the start of the M-augmenting path and the unmatched worker at the end of the M-augmenting path will both be matched at the expense of one match in the middle. Comparing the two paths in the second panel of Figure 3 illustrates this. The matching

$M = \{1 - B, 2 - C\}$ in an M-augmenting path can always be increased by *switching* the dashed and solid links resulting in an extra link, i.e., $M = \{A - 1, B - 2, C - 4\}$.

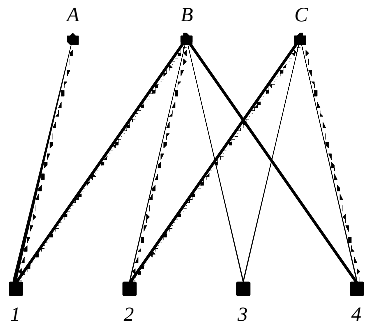

Figure 4: Symmetric difference operation ($N \Delta M$)

What remains to be shown is that if a matching M has no M-augmenting paths, it is a maximum matching. This can be proven by contradiction. Suppose that in a particular graph in our setting there is a matching M for which there are no M-augmenting paths but that (contrary to Berge's Theorem) this matching is not a maximum matching. Then there is a matching N (i.e., $A - 1, B - 2, C - 4$; dashed lines in Figure 4) with more links than M (i.e., $1 - B, 2 - C$; dotted lines in Figure 4), $|N| > |M|$. Now consider the symmetric difference $N \Delta M$ defined as the set of links that is either in N or M but not in both (the sum of dashed and dotted lines in Figure 4, $A-1$, $B-2$, $C-4$, $1-B$, $2-C$). Each worker or firm can have at most 2 links in $N \Delta M$ because he is hired by at most one firm in M and at most one firm in N. Moreover, the links of the paths alternate between being in M and being in N, because by the definition of a matching, no node can have two links in M or two links in N. Since by assumption N is strictly bigger than M there must be at least one path in $N \Delta M$ with an odd number of links that starts with a firm (worker) in N and ends with a worker (firm) in N (i.e., $A - 1 - B - 2 - C - 4$). But then this is an M-augmenting path because the firm and worker at the start and end of the path are (by the symmetric difference operation) not in M. This gives us the desired contradiction, because we started by assuming that M has no M-augmenting paths.

Thus, in order to show that ex post competition leads to a maximum matching we need to rule out that an M-augmenting path exists.

4.2.2 Optimality of ex post competition

In this section we show that for a given network, ex post competition with complete recall generates a maximum matching.

Lemma 3 *If a firm remains unmatched after T rounds, then all workers along an M-alternating path that starts with the unmatched firm must earn a wage equal to the marginal product, i.e., $w = 1$.*

Proof: To see why all workers along the M-alternating path receive $w = 1$, first note that if a firm with candidates (firm A) remains unmatched after T rounds, then all its applicants must have accepted a wage $w = 1$ (since if at least one of its candidates would earn $w < 1$, firm A would have offered that worker $w < 1$ and make positive profits). But then the other candidate of the next firm along the M-alternating path (firm B) that hired firm A's candidate must also receive $w = 1$ otherwise firm B would have hired that worker at a $w < 1$. Repeating this argument implies that all firms along the M-alternating path pay a wage of 1. ■

Lemma 3 implies that all workers in M-alternating paths that start with an unmatched firm have been offered a wage equal to 1 in round T.

Lemma 4 *If a worker remains unmatched after T rounds, each firm along an M-alternating path that starts with the unmatched worker pays no more than $w = \Delta$.*

Proof: The firm (firm A) to which the unmatched worker (worker 1) applied will offer the worker who it hired (worker 2) at most $w = 0$ otherwise it could have offered the job to the unmatched worker 1. But then the worker (worker 3) who is hired by the next firm along the M-alternating path (firm B) must also earn $w \in \{0, \Delta\}$, else this firm (B) would have hired worker 2 by offering $w = \Delta$. Repeating this argument implies that all firms along the M-alternating path that starts with an unmatched worker pay a wage $w \in \{0, \Delta\}$. ■

Lemma 4 implies that all workers in M-alternating paths that start with an unmatched worker have been offered a wage $w \in \{0, \Delta\}$ in round T. According to Berge's Theorem a maximum matching exists if and only if there is no M-alternating path that starts with an unmatched worker and ends with an unmatched firm, i.e., if and only if there is no M-augmented path. Given the wage pattern in an M-alternating path that starts with an unmatched worker (Lemma 4) or with an unmatched firm (Lemma 3), we can write down our main Proposition.

Proposition 2 *Ex post competition leads to a maximum matching in any realized network.*

Proof: Suppose it would not lead to a maximum matching. In that case there would exist an M-augmenting path with at least one unmatched worker and one unmatched firm. But then Lemmas 3 and 4 imply that all firms along the M-augmenting path (that is also an M-alternating path) offer both a wage $w \in \{0, \Delta\}$ and a wage equal to 1, which is a contradiction. ■

Note that this result is very general. If firms can only interview a subset of their workers as in Wolthoff (2011) or one as in Albrecht et al. (2006) and Galenianos and Kircher (2009), the realized network will be different but Proposition 2 still holds. The same is true, if workers have for example different search costs and consequently send out different numbers of applications. Also, if firms can create shortlists of at most n candidates, our result holds. This just requires an intermediate step where all firms with more than n candidates must eliminate (at random) the exceeding number of links. After this intermediate step, a new network arises for which the same results on maximum matching hold as above.

The flexibility of ex post competition in wages is central to achieve efficiency in network clearing. If firms commit ex ante to a posted wage and do not adjust their wages ex post, we can typically observe different wages along an M-alternating path. If both end nodes of the M-alternating path are unmatched, i.e., if we have an M-augmenting path, there is no mechanism inherent in the matching process associated with wage commitment that can induce the matched firm-worker pairs to rematch with the unmatched firm and worker at the end of the M-augmenting path. Thus, the inefficient network clearing result of wage commitment from the 3 by 3 example of section 3 holds in general. Therefore, Berge's Theorem also implies the following Corollary.

Corollary 1 *If firms commit not to increase their posted wages ex post, network clearing is generally inefficient and the maximum matching is not realized.*

Corollary 1 shows that directed search models with fixed posted wages are not able to solve the second coordination friction (firms do not know which workers are considered by other firms). Thus, although directed search with fixed posted wages is constraint efficient in terms of firm entry and number of applications that workers send, see Kircher (2009), it generally does not generate the maximum matching that is possible given the network that is formed between firms and their applicants.[8]

5 Random network formation

In this section we use the graph theoretical results in Frieze and Melsted (2012) to derive the expected number of matches $M(v, u, a)$ that results from maximum matching on the realized network and random search. We only consider large markets, i.e. $v, u \to \infty$ with $\theta = v/u$.

5.1 The matching function

Point of departure is the Karp-Sipser algorithm that ensures a maximum matching in large markets (see Karp-Sipser, 1981). The algorithm consists of two phases. Phase 1 is based on the fact that if a vacancy has only one applicant, then there exists a maximum matching that includes this vacancy and its applicant.[9] Thus, in any round of Phase 1 the algorithm picks a vacancy with only one application and allocates the respective worker-firm pair to the matching. This part is repeated as long as there are vacancies with one application. Note, that vacancies that initially belong to the class of that has more than one applicant can later in the process become part of the class of vacancies with only one application. The number of matches generated in Phase 1 of the algorithm

[8]Note, that Kircher's (2009) equilibrium is constrained efficient because the planner takes the existence of a subset of firms that match first as given, whereas here this is not part of the planner's constraint.

[9]Denote the maximum matching on a realized network (graph) G by M. If M' is the maximum matching on $G \setminus \{u, v\}$,i.e on the graph without the vacancy v (of degree one) and its applicant u, then M includes the firm-worker pair $\{u, v\}$.

equals the number of rounds. The round in which Phase 1 ends is denoted by t^*. The number of unemployed workers at the end of Phase 1 is thus given by $u(t^*) = u - t^*$.

The graph that remains has by construction of the Karp-Sipser algorithm a minimum degree of two, i.e., each vacancy has at least two applications and each remaining worker has still a applications.[10] Frieze and Melsted (2012) show for a large market that the expected number of maximum matches on the remaining graph equals the minimum of the remaining vacancies or remaining workers, i.e., $\min[v(t^*), u(t^*)]$ as $v, u \to \infty$. Thus, the expected number of maximum matches on the initial graph is given by,

$$M(v, u, a) = t^* + \min[v(t^*), u(t^*)].$$

There are two cases to consider. In case 1 all unemployed workers are matched, either because $\min[v(t^*), u(t^*)] = u(t^*)$, which implies $M(v, u, a) = t^* + u(t^*) = u$, since $u(t^*) = u - t^*$, or because Phase 1 of the Karp-Sipser algorithm continues until the last worker is matched, i.e., $t^* = u$. In case 2, the expected number of matches is equal to $M(v, u, a) = t^* + v(t^*)$, since $\min[v(t^*), u(t^*)] = v(t^*)$. Following Frieze and Melsted (2012) we will provide the conditions that determine which case prevails after we derived the matching function.

5.1.1 Phase 1 of the Karp-Sipser algorithm

Phase 1 of the Karp-Sipser algorithm can be described by a sequence of Poisson distributions. Since search is random, the initial distribution of applications y_j at vacancy j follows a Poisson distribution with mean $z_0 = au/v$, i.e.,

$$P(y_j = k|z_0) = \frac{(z_0)^k e^{-z_0}}{k!}.$$

The number of applications at the remaining vacancies with two or more applications in round t still follows a Poisson distribution albeit with a different mean. This can be understood as follows. First, note that in each round in Phase 1 exactly one worker is matched and all other workers and applications remain unaffected. This implies that the

[10] Each remaining worker has still all a applications, because in Phase 1 the algorithm removes only vacancies with one applicant, i.e., only the linked worker is matched and removed but no other worker is affected.

applications of the remaining workers are still randomly distributed. Second, with every matched worker, $a-1$ applications are withdrawn randomly from the remaining vacancies. It follows that the number of applications at the remaining vacancies with two or more applications (type 2) follow a truncated Poisson distribution (see Frieze and Melsted, page 5, 2012), i.e.,

$$P(y_j = k|z_t, k \geq 2) = \frac{z_t^k e^{-z_t}}{k!\,(1 - e^{-z_t} - z_t e^{-z_t})}.$$

The number of applications going to vacancies with only one applicant (type 1) does not follow the same truncated Poisson distribution because the number of vacancies with one applicant is reduced by one each round.

The parameter z_t that governs the underlying Poisson at the type 2 vacancies is simply the average number of applications going to vacancies with at least two applications. Let v_t^1 denote the number of vacancies with exactly one application, and v_t denote the number of vacancies with at least two applications. Since au_t equals the total number of applications in round t, we can write,

$$\frac{au_t - v_t^1}{v_t} = \sum_{k=2}^{\infty} k \frac{z_t^k e^{-z_t}}{k!\,(1 - e^{-z_t} - z_t e^{-z_t})},$$

which implies,

$$\frac{au_t - v_t^1}{v_t} = \frac{z_t\,(1 - e^{-z_t})}{1 - e^{-z_t} - z_t e^{-z_t}}. \tag{1}$$

Note that for given u_t, v_t^1, and v_t the solution z_t to the function defined in equation (1) is unique.[11]

The development of the number of type-1 vacancies, v_{t+1}^1 and of the number vacancies with two and more applications, v_{t+1} given the set $\Omega_t = \{u_t, v_t^1, v_t\}$ can -according to the Karp-Sipser algorithm- be described by the following difference equations,

$$E\left[v_{t+1}^1 - v_t^1 | \Omega_t\right] = -1 - \frac{(a-1)}{au_t} v_t^1 + \frac{(a-1)}{au_t} \frac{(z_t)^2 e^{-z_t}}{1 - e^{-z_t} - z_t e^{-z_t}} v_t, \tag{2}$$

$$E\left[v_{t+1} - v_t | \Omega_t\right] = -\frac{(a-1)}{au_t} \frac{(z_t)^2 e^{-z_t}}{1 - e^{-z_t} - z_t e^{-z_t}} v_t. \tag{3}$$

The type-1 vacancies,v_t^1 decrease by 1 in each round. Furthermore, as the remaining $a-1$ applications of the matched worker are withdrawn an additional vacancy with exactly

[11] This can easily be seen by rearranging equation (1) as follows, $-z_t\gamma_t = (z_t - \gamma_t)(e^{z_t} - 1)$, where $\gamma_t = (au_t - v_t^1)/v_t$.

one applicant is eliminated with probability $(a-1)/au_t$ and a vacancy with exactly two applicants becomes a type 1-vacancy with probability $2(a-1)/au_t$. Note, that the term $(z_t)^2 e^{-z_t}/(1-e^{-z_t}-z_t e^{-z_t})$ equals the probability that a type-2 vacancy, v_t has exactly two applications. The vacancies with two applicants that become type-1 vacancies reduce the number of type-2 vacancies.

Frieze and Melsted (2012) show that the difference equations (2) and (3) can in a large market be approximated by the following differential equations,[12]

$$\frac{dv^1(t)}{dt} = -1 - \frac{(a-1)}{au(t)} v^1(t) + \frac{(a-1)}{au(t)} \frac{z(t)^2 e^{-z(t)}}{1-e^{-z(t)}-z(t)e^{-z(t)}} v(t),$$
$$\frac{dv(t)}{dt} = -\frac{(a-1)}{au(t)} \frac{z(t)^2 e^{-z(t)}}{1-e^{-z(t)}-z(t)e^{-z(t)}} v(t),$$

where $u(t) = U - t$ and $z(t)$ satisfies equation (1) and the Boundary conditions are given by $u(0) = u$, $z(0) = au/v$, $v^1(0) = vz(0)e^{-z(0)}$, and $v(0) = v\left(1-e^{-z(0)}-z(0)e^{-z(0)}\right)$. The solution to this differential equation system is given by the following Lemma.

Lemma 5 *The solution to the differential equations is*

$$t = u\left(1-\left(\frac{z(t)}{z(0)}\right)^{\frac{a}{a-1}}\right), \tag{4}$$
$$u(t) = u\left(\frac{z(t)}{z(0)}\right)^{\frac{a}{a-1}}, \tag{5}$$
$$v(t) = v\left(1-e^{-z(t)}-z(t)e^{-z(t)}\right), \tag{6}$$
$$v^1(t) = vz(t)\left(\left(\frac{z(t)}{z(0)}\right)^{\frac{1}{a-1}} + e^{-z(t)} - 1\right), \tag{7}$$

where

$$\frac{au(t)-v^1(t)}{v(t)} = \frac{z(t)\left(1-e^{-z(t)}\right)}{1-e^{-z(t)}-z(t)e^{-z(t)}}. \tag{8}$$

Proof: See Frieze and Melsted (2009) Lemma 9, which is summarized in Appendix A.4.

Phase 1 ends in round t^* where all vacancies with only one application are withdrawn, i.e. at $v^1(t^*) = 0$. The respective $z(t^*)$ is given by the largest non-negative solution to,

$$\left(\frac{z(t^*)}{z(0)}\right)^{\frac{1}{a-1}} = 1 - e^{-z(t^*)}. \tag{9}$$

[12] Luby et al. (2001) and Dembo and Montanari (2008) were the first who showed that the algorithm can be approximated by two differential equations.

The expected number of matches during Phase 1 is therefore given by,

$$t^{*}=u\left(1-\left(1-e^{-z(t^{*})}\right)^{a}\right). \tag{10}$$

The intuition behind the expected number of matches during Phase 1 is the following. Consider a particular worker, who sent a applications. With probability $e^{-z(t^{*})}$ there were no other applicants at the firm where he sent a particular application. With probability $\left(1-e^{-z(t^{*})}\right)$ there were other applicants. With probability $\left(1-e^{-z(t^{*})}\right)^{a}$ there were applicants at all the firms where he applied and with probability $\left(1-\left(1-e^{-z(t^{*})}\right)^{a}\right)$ there was at least one firm where he had applied to that had no other applicants. The worker was therefore the only applicant and matched for sure during Phase 1. Since there are u workers, $u\left(1-\left(1-e^{-z(t^{*})}\right)^{a}\right)$ is the expected number of workers that got matched during Phase 1.

All unemployed workers are matched during Phase 1, if there exists a round t such that $u(t)=0$ and $v^{1}(t)>0$. The threshold value $z_{1}(0)$ that ensures that $v^{1}(t)>0$ holds for all $z(0)\leq z_{1}(0)$ is given by rearranging equation (7) at $v^{1}(t)=0$, i.e.,

$$z_{1}(0)=\frac{z_{1}(t)}{\left(1-e^{-z_{1}(t)}\right)^{(a-1)}}, \tag{11}$$

where $z_{1}(t)$ is no longer given by equation (9), but is defined such that the right hand side of equation (9) touches the left hand side of equation (9), i.e., by,

$$\frac{\partial}{\partial z_{1}(t)}\left(\frac{z_{1}(t)}{z_{1}(0)}\right)^{\frac{1}{a-1}}=\frac{\partial\left(1-e^{-z_{1}(t)}\right)}{\partial z_{1}(t)}\Longrightarrow\frac{1}{a-1}\left(\frac{z_{1}(t)}{z_{1}(0)}\right)^{\frac{1}{a-1}}\frac{1}{z_{1}(t)}=e^{-z_{1}(t)}.$$

The right hand side of equation (9) touches the left hand side of equation (9), i.e., they do not cross, because $v^{1}(t)=0$ is not reached at a finite t but only in the limit as $t\rightarrow\infty$.

Rearranging using the definition of the threshold (11) implies that $z_{1}(t)$ is the positive solution to equation,

$$\frac{1-e^{-z_{1}(t)}}{(a-1)\,e^{-z_{1}(t)}}=z_{1}(t). \tag{12}$$

Lemma 6 *If $au/v\leq z_{1}(0)$, then $M(v,u,a)=u$.*

Proof: See Theorem 2 in Frieze and Melsted (2009).

For $a=2$ this condition is never satisfied, for $a=3$ it is satisfied if $v\gtrsim 3u$ and for $a=5$ if $v\gtrsim 2u$. These examples show that the number of vacancy need to be relatively high in order to ensure that all unemployed workers are matched. In section 6 we show that in the case where vacancy creation is costly the socially optimal number of vacancies implies $au/v > z_1(0)$.

5.1.2 The expected number of matches after Phase 2

If Phase 1 ends before all unemployed workers are matched, i.e., if $au/v = z(0) > z_1(0)$, then Frieze and Melsted (2012) show that the total number of matches equals $M(v, u, a) = t^* + \min[v(t^*), u(t^*)]$.[13] The expected number of vacancies and unemployed workers on the remaining graph are according to Lemma 5 given by

$$u(t^*) = u\left(1 - e^{-z(t^*)}\right)^a, \text{ and } v(t^*) = v\left(1 - e^{-z(t^*)} - z(t^*)\, e^{-z(t^*)}\right).$$

The total number of matches equals the number of unemployed workers, if $u(t^*) = \min[v(t^*), u(t^*)]$. The threshold value $z_2(0)$ that ensures $u(t^*) \leq v(t^*)$ for all $z(0) \leq z_2(0)$ is defined by using equation (9),

$$z_2(0) = \frac{z_2(t^*)}{\left(1 - e^{-z_2(t^*)}\right)^{(a-1)}}, \tag{13}$$

where $z_2(t^*)$ is determined by $u(t^*) = v(t^*)$. Rearranging implies,

$$a = \frac{z_2(t^*)\left(1 - e^{-z_2(t^*)}\right)}{1 - e^{-z_2(t^*)} - z_2(t^*)\, e^{-z_2(t^*)}}. \tag{14}$$

The threshold value $z_2(0)$ can be below or above the threshold value $z_1(0)$. This implies that the expected number of matches can be summarized by the following Lemma.

Lemma 7 *If* $au/v \leq \max[z_1(0), z_2(0)]$, *then* $M(v, u, a) = u$. *If* $au/v > \max[z_1(0), z_2(0)]$ *then*

$$M(v, u, a) = u\left(1 - \left(1 - e^{-z(t^*)}\right)^a\right) + v\left(1 - e^{-z(t^*)} - z(t^*)\, e^{-z(t^*)}\right). \tag{15}$$

Proof: This follows from Theorem 3 in Frieze and Melsted (2009) for $a \geq 3$ and Devroye and Morin (2003), Pagh and Rodler (2004) and Kutzelnigg (2006) for $a = 2$. For $a = 1$ note that equation (9) implies $z(t^*) = z(0) = u/v$. Inserting into the matching function (15) implies $M(v, u, 1) = v\left(1 - e^{-z(0)}\right)$ which is the well known urn-ball matching function.

[13] Theorem 1 in Frieze and Melsted (2012) is based on $a \geq 3$. The case of $a = 2$ is dealt with in Devroye and Morin (2003), Pagh and Rodler (2004) and Kutzelnigg (2006).

5.2 Efficient network formation

In order to show that random network formation is constraint efficient, we allow the social planner to choose the set of firm subgroups C (where each subgroup c is defined by a certain color), the measure of vacancies v_c within each subgroup c and the probability p_c that a worker sends a particular application to subgroup $c \in C$. Note, that $\sum_{c \in C} v_c = v$ and $\sum_{c \in C} p_c = 1$. Within each subgroup c the applications at each vacancy are distributed according to the respective Poisson distribution,

$$P\left(y_j = k | z_0^c\right) = \frac{\left(z_0^c\right)^k e^{-z_{0,c}}}{k!},$$

with parameter $z_{0,c} = p_c au/v_c$. The parameter $z_{t,c}$ that governs the underlying Poisson distribution for vacancies with at least two applications must satisfy,

$$\frac{p_c au_t - v_{t,c}^1}{v_{t,c}} = \frac{z_{t,c}\left(1 - e^{-z_{t,c}}\right)}{1 - e^{-z_{t,c}} - z_{t,c} e^{-z_{t,c}}}. \tag{16}$$

The development of the number of vacancies with exactly one application $v_{t+1,c}^1$ and the number vacancies with two and more applications $v_{t+1,c}$ given the set $\Omega_{t,c} = \left\{u_t, v_{t,c}^1, v_{t,c}, p_c\right\}$ in round t can according to the Karp-Sipser algorithm be described by the following difference equations for all $c \in C$,

$$E\left[v_{t+1,c}^1 - v_{t,c}^1 | \Omega_{t,c}\right] = -\kappa_{t,c} - \frac{p_c a - \kappa_{t,c}}{p_c au_t} v_{t,c}^1 + \frac{p_c a - \kappa_{t,c}}{p_c au_t} \frac{\left(z_{t,c}\right)^2 e^{-z_{t,c}}}{1 - e^{-z_{t,c}} - z_{t,c} e^{-z_{t,c}}} v_{t,c},$$

$$E\left[v_{t+1,c} - v_{t,c} | \Omega_{t,c}\right] = -\frac{p_c a - \kappa_{t,c}}{p_c au_t} \frac{\left(z_{t,c}\right)^2 e^{-z_{t,c}}}{1 - e^{-z_{t,c}} - z_{t,c} e^{-z_{t,c}}} v_{t,c},$$

where $\kappa_{t,c}$ denotes the probability that the randomly picked vacancy with only one applicant in round t is a vacancy of subgroup c, i.e., $\kappa_{t,c} = v_{t,c}^1 / \sum_{c \in C} v_{t,c}^1$. The remaining $a - 1$ applications are going with probability $\left(p_c a - \kappa_{t,c}\right) / p_c au_t$ to a vacancy of subgroup c with exactly one applicant. The expected number of applications going to vacancies of subgroup c are given by $p_c a - \kappa_{t,c} = \kappa_{t,c}\left(p_c a - 1\right) + \left(1 - \kappa_{t,c}\right) p_c a$, i.e., if a vacancy with one applicant of subgroups c is removed, then there is an expected number of $\left(p_c a - 1\right)$ applications still going to subgroup c and if a vacancy with one applicant of a different subgroup is removed, which happens with probability $1 - \kappa_{t,c}$, then there is an expected number of $p_c a$ applications still going to subgroup c. The remaining parts are identical to the random application case.

The solution to the respective differential equation system is given by the following Lemma.

Lemma 8 *The solution to the differential equations is*

$$t = u\left(1 - \left(\prod_{c\in C}\left(\frac{z_c(t)}{z_c(0)}\right)^{p_c}\right)^{\frac{a}{a-1}}\right), \tag{17}$$

$$u(t) = u\left(\prod_{c\in C}\left(\frac{z_c(t)}{z_c(0)}\right)^{p_c}\right)^{\frac{a}{a-1}}, \tag{18}$$

$$v_c(t) = v_c\left(1 - e^{-z_c(t)} - z_c(t)\, e^{-z_c(t)}\right) \text{ for all } c\in C, \tag{19}$$

$$v_c^1(t) = v_c\left(z_c(0)\left(\prod_{c\in C}\left(\frac{z_c(t)}{z_c(0)}\right)^{p_c}\right)^{\frac{a}{a-1}} + z_c(t)\left(e^{-z_c(t)} - 1\right)\right) \text{ for all } c\in C, \tag{20}$$

where

$$\frac{p_c a u(t) - v_c^1(t)}{v_c(t)} = \frac{z_c(t)\left(1 - e^{-z_c(t)}\right)}{1 - e^{-z_c(t)} - z_c(t)\, e^{-z_c(t)}}. \tag{21}$$

Proof: See Appendix A.5.

Note, that Phase 1 of the Karp-Sipser algorithm ends when in all subgroups $c \in C$ the number of vacancies with one applicant is equal to zero, i.e., $v_c^1(t^*) = 0$ for all $c \in C$. This is guaranteed by the definition of $\kappa_c(t) = v_c^1(t) / \sum_{c\in C} v_c^1(t)$.

The expected number of matches is either equal to the number of unemployed workers or to the expected number of rounds in Phase 1 plus the expected number of matches on the remaining graph. In the first case, having different subgroups with different application probabilities generates the same expected number of matches as the case where workers randomize over all vacancies. In the second case, the result that the expected number of matches on the remaining graph is equal to $\min[u(t^*), v(t^*)]$ requires full randomization over vacancies. In order to prove that full randomiztion over vacancies is socially efficient, we only need to show that $t^* + \min\left[u(t^*), \sum_{c\in C} v_c(t^*)\right]$ is maximized if all vacancies have an equal probability to receive an application, i.e., at $p_c/v_c = 1/v$ for all $c \in C$. This is guaranteed if all vacancies are equally distributed across subgroups and if all subgroups are equally likely to receive an application, i.e., at $v_c = v/n$ and $p_c = 1/n$ for all $c \in C$, where n equals the number of subgroups.

Proposition 3 *Random search is socially efficient, i.e.,* $t^* + \min\left[u(t^*), \sum_{c \in C} v_c(t^*)\right]$ *is maximized if*

$$v_c = \frac{v}{n} \text{ and } p_c = \frac{1}{n} \text{ for all } c \in C.$$

Proof: See Appendix A.6.

Finally, we conjecture that under directed search network efficiency will also be efficient. From the competing mechanism literature, i.e. McAfee (1993), Peters and Severinov (1997) and Albrecht et al. (2013), we know that when search is non rival, that in a large market, the surviving mechanisms are simple auctions without entry fees and reserve prices. In our setting, the posted wage would play the same role as a reserve price. The intuition for why reserve prices go to zero is that only efficient mechanisms survive that leave no surplus on the table. Since buyers can vote with their feet, sellers must offer the buyers market-utility and they can keep the rest. A positive reserve price implies a lower arrival rate of buyers and this is inefficient. Lowering the reserve price increases profits. For similar reasons we conjecture that in a directed-search equilibrium, lowering the posted wage to zero is the most efficient way to offer workers the required market utility. A formal proof goes beyond the scope of this paper. If all firms post a zero wage, workers will again randomize over all vacancies and as we showed above this is desirable from a social point of view.

6 Search intensity and entry

Although random search with ex post competition in wages is efficient in terms of network formation and network clearing, it need not be efficient in other dimensions like vacancy creation, worker participation or the number of applications that workers send out. In this section we derive the conditions for optimal firm and worker entry and for the optimal number of applications per worker.

6.1 Efficient search intensity

Since workers' payoffs are independent of whether they are in a worker or even subgraph (if firms receive the full surplus in even subgraphs), they do not take into account that an additional application can turn a worker subgraph into an even subgraph and thereby generate an additional match. There is also a rent seeking externality caused by the fact that an additional application can increase the chances of turning an even subgraph into a firm subgraph. This does not add any social value because no additional match is created, all it does is redistributing surplus from the firm to the worker. The following Proposition shows result that social efficiency requires that workers receive the full surplus in even subgraphs (i.e they receive the worker-optimal point in the core). Note that Kim and Kircher (2012) derived a similar result in a model where workers send one application only.

Proposition 4 *Workers choose the efficient number of applications a, if workers receive the full surplus in even subgraphs.*

Proof: Consider the social and private returns of an additional application. We denote the social return by $S^{i \to j}$, where the superscript i denotes the subgraph where the worker would end up if he would not have sent the additional application, $i \in \{f, w, e\}$, and $j \in \{f, w, e\}$ denotes the subgraph of the firm that received the application if the worker would not have sent the additional application. The private return is similarly denoted by $P_k^{i \to j}$, where the subscript $k \in \{f, w\}$ denotes whether firms (f) or workers (w) receive the full surplus in even subgraphs. Table 1 summarizes all possible cases.

Note first that an additional match is only generated if $i = w$ and $j = f$. The reason is simply that in all other cases either the worker or the firm would match anyway.

The private returns $P_k^{i \to j}$ depend on whether firms $(k = f)$ or workers $(k = w)$ receive the full surplus in even subgraphs. If $k = f$ a worker only benefits, if $i \in \{w, e\}$, and $j = f$. If $i = w$, and $j = f$, the additional application will make the worker either part of an even or firm subgraph. This can simply be proven by contradiction. Suppose that the worker remains part of the worker subgraph. In this case a worker in a worker subgraph would have to be linked to a firm in a firm subgraph. This is ruled out by the Decomposition Theorem of Corominas-Bosch (2004), i.e., weak nodes are only linked to strong nodes. The expected private return $P_f^{w \to f}$ depends on the probability that the worker ends up in a firm subgraph, i.e., $P_f^{w \to f} \in (0, 1)$. If the worker would be part of an

Table 1: Social and private returns of an additional application

application from i to j subgraph	social return	private return (change in wages)	
$i \to j$	$S^{i \to j}$	$P_f^{i \to j}$	$P_w^{i \to j}$
$f \to f$	0	0	0
$f \to w$	0	0	0
$f \to e$	0	0	0
$w \to f$	1	$(0, 1)$	1
$w \to w$	0	0	0
$w \to e$	0	0	0
$e \to f$	0	1	0
$e \to w$	0	0	0
$e \to e$	0	0	0

even subgraph without the additional application, i.e., $i = e$, and sends the application to a firm in a firm subgraph, i.e., $j = f$, the application will always ensure that the worker becomes part of the firm subgraph. The reason is that the Decomposition Theorem of Corominas-Bosch (2004) rules out that a firm in a firm subgraph is linked to a worker that is not part of a firm subgraph (compare section 4.1.1).

If $k = w$, a worker only benefits, if $i = w$, and if $j \in \{f, e\}$. If $i = w$ and $j = f$, the additional application will make the worker either part of an even or firm subgraph as shown above. In both cases the worker's wage will increase from zero to one. If $i = w$ and $j = e$, the firm in the even subgraph will become part of the worker subgraph, i.e., the worker will stay in the worker subgraph and earn a wage of zero. The reason is that the Decomposition Theorem of Corominas-Bosch (2004) rules out that a firm in an even subgraph is linked to a worker in a worker subgraph (compare section 4.1.1). Thus, private and social returns are always aligned if workers receive the full surplus in an even subgraph. ■

This suggests that our wage game where firms offer wages is not socially efficient. Workers apply to too many firms in order to prevent ending up in an even subgraph. The optimal outcome can be decentralized by reversing the role of workers and firms in our wage game.

6.2 Entry efficiency

Vacancy creation or worker participation need not be efficient due to the fact that firms do not internalize that when they enter they also destroy profits of other firms by making it more likely that these firms end up in firm subgraphs. Similarly, workers do not take into account that when they enter they make it more likely that other workers end up in a worker subgraph.

First consider vacancy creation. Denote the cost of creating a vacancy by $h > 0$. In the decentralized market vacancies are created until the expected return is equal to the cost of vacancy creation, i.e.,

$$\frac{M(v,u,a)}{v}\left(1 - E[w]\right) = h, \tag{22}$$

where $E[w]$ equals the expected wage paid to workers. The socially optimal number of vacancies in the market is then given by equating the marginal output (matches) generated by an additional vacancy to the cost of creating this vacancy, i.e.,

$$M'_v(v,u,a) = h. \tag{23}$$

Note, that the derivative of the matching function is equal to zero, if $M(v,u,a) = u$ and equal to,

$$\begin{aligned} M'_v(v,u,a) &= v\left(z(t^*)\,e^{-z(t^*)} - \frac{au}{v}\left(1 - e^{-z(t^*)}\right)^{a-1} e^{-z(t^*)}\right)\frac{dz(t^*)}{dv} \\ &\quad + \left(1 - e^{-z(t^*)} - z(t^*)\,e^{-z(t^*)}\right), \\ &= \left(1 - e^{-z(t^*)} - z(t^*)\,e^{-z(t^*)}\right), \end{aligned} \tag{24}$$

if the matching function is given by equation (15).[14] The cost of creating a vacancy already implies the following bounds on the optimal number of vacancies.

Lemma 9 *If the cost of creating a vacancy is positive, i.e., $k > 0$, the socially optimal number of vacancies satisfies,*

$$v < \min\left[\frac{au}{z_1(0)}, \frac{au}{z_2(0)}\right].$$

[14] Note that the last equality in equation (24) follows from equation (9).

Proof: Since $v \geq \min[au/z_1(0), au/z_2(0)]$ implies $M(v,u,a) = u$, it follows that $M'_v(v,u,a) = 0$, which violates the social optimality condition (23). ■

Lemma 9 implies that it is never efficient to increase the number of vacancies such that all unemployed workers are matched. Instead it is efficient to leave some workers unmatched and save on vacancy creation costs. Thus, the economy is only socially efficient, if the matching function is given by equation (15).

Combining equations (22) to (24) and using the socially efficient matching function given by equation (15) implies that free entry of vacancies is socially optimal, if

$$1 - E[w] = \frac{v\left(1 - e^{-z(t^*)} - z(t^*)\,e^{-z(t^*)}\right)}{u\left(1 - (1 - e^{-z(t^*)})^a\right) + v\left(1 - e^{-z(t^*)} - z(t^*)\,e^{-z(t^*)}\right)}. \tag{25}$$

Next, consider the participation condition of workers. Suppose that the economy is populated by a mass, n, of workers who differ in their participation cost x. Rank workers according to their cost of entry and let the cumulative distribution function $H(x)$ be continuous. The number workers participating in the market, i.e., the number of unemployed workers, is given by $u = nH\left(x^P\right)$. The entry cost of the marginal worker x^P, who is indifferent between participating in the market or staying out, is given by,

$$\frac{M(v,u,a)}{u}E[w] = x^P. \tag{26}$$

The social planner, however, trades off the marginal output created by an additional worker with his/her cost of entry, i.e.,

$$M'_u(v,u,a) = x^S. \tag{27}$$

The derivative of the (socially efficient) matching function, given in equation (15), is equal to,

$$\begin{aligned} M'_u(v,u,a) &= \left(1 - \left(1 - e^{-z(t^*)}\right)^a\right) \\ &\quad + \left(vz(t^*)\,e^{-z(t^*)} - ua\left(1 - e^{-z(t^*)}\right)^{a-1} e^{-z(t^*)}\right)\frac{dz(t^*)}{du}, \\ &= \left(1 - \left(1 - e^{-z(t^*)}\right)^a\right). \end{aligned} \tag{28}$$

Combining equations (26) to (28) implies that the participation condition is efficient, if $x^P = x^S$, or

$$E[w] = \frac{u\left(1 - \left(1 - e^{-z(t^*)}\right)^a\right)}{u\left(1 - (1 - e^{-z(t^*)})^a\right) + v\left(1 - e^{-z(t^*)} - z(t^*)\,e^{-z(t^*)}\right)}. \tag{29}$$

Note, that the efficiency conditions for vacancy and worker entry are identical.

Note that equation (29) simplifies for $a = 1$, i.e.,

$$E[w|a=1] = \frac{z(0)\, e^{-z(0)}}{1 - e^{-z(0)}}, \tag{30}$$

since equation (9) implies $z(t^*) = z(0) = u/v$ for $a = 1$. This is the wage in Burdett, Shi and Wright (2001). Thus, worker participation and firm entry is efficient, if the expected wage $E[w|a=1]$ is equal to the ratio of vacancies with exactly one application, i.e., $vz(0)\, e^{-z(0)}$, to all matched vacancies, i.e., $v\left(1 - e^{-z(0)}\right)$. Since all vacancies with one application are by definition in even subgraphs and since all applications with at least two applications are by definition in worker subgraphs, it follows that vacancy creation and worker participation are only efficient, if workers receive the full surplus in even subgraphs. This result is equivalent to the result by Kim and Kircher (2012), who show in a model with $a = 1$ that entry is efficient, if workers are awarded the worker-maximizing point in the core.[15] The following Proposition shows that this efficiency result holds for any number of applications.

Proposition 5 *(i) Worker participation and vacancy creation are socially optimal, if workers receive the full surplus in even subgraphs.*
(ii) Worker participation is inefficiently low and vacancy creation inefficiently high, if firms receive the full surplus in even subgraphs.

Proof: Consider first the social and private return of a new worker. We denote the social return by S^y, where the superscript y denotes the set of subgraphs in which the firms that received one of the a applications of the new worker would have ended up if the new worker did not enter. In other words, the new worker applies to a firms and those firms would in the absence of the new worker belong to different (possibly the same) subgraphs. The set of possible subgraphs is given by $y \in \{f, w, e, \{f, w\}, \{f, e\}, \{w, e\}, \{f, w, e\}\}$, where f, w and e denote a firm, worker and even subgraph. The private return of the worker is denoted by P_k^y, where the subscript $k \in \{f, w\}$ denotes whether firms (f) or

[15] Benoit, Kennes, and King (2006) show that it is equivalent to the Mortensen (1982) rule, if workers receive the full surplus in case they are the only candidate.

workers (w) receive the full surplus in even subgraphs. Table 2 summarizes all possible cases.

Table 2: Social and private returns of a new worker

applications of the entrant are send to y subgraphs	social return S^y	private return (wages) P_f^y	P_w^y
$y = f$	1	$(0, 1)$	1
$y = w$	0	0	0
$y = e$	0	0	0
$y = \{f, w\}$	1	$(0, 1)$	1
$y = \{f, e\}$	1	$(0, 1)$	1
$y = \{w, e\}$	0	0	0
$y = \{f, w, e\}$	1	$(0, 1)$	1

Note first that an additional match is only generated, if $y \in \{f, \{f, w\}, \{f, e\}, \{f, w, e\}\}$. The reason is simply that in all other cases all contacted firms are already matched, i.e., if firms are part of a worker or even subgraph.

The private returns of a new worker, P_k^y, depend on whether firms ($k = f$) or workers ($k = w$) receive the full surplus in even subgraphs. If $k = f$ the new entrant only benefits, i.e., is paid a wage equal to one, if she/he becomes part of a firm subgraph. This is only possible, if $y \in \{f, \{f, w\}, \{f, e\}, \{f, w, e\}\}$ and if the contacted firm remains in the firm subgraph. This probability need not be one, since there exists a certain chance that the new entrant turns (part of) the previous firm subgraph into an even subgraph (e.g. if the firm subgraph has exactly one excess vacancy). This also implies that the private return P_k^y is only equal to one if an additional match is created and workers receive the full surplus in even subgraphs, i.e., $k = w$.

If the entrant contacts only firms in $y = w$ subgraphs, the worker will become part of a worker subgraph and receives a wage equal to zero. If the entrant contacts only firms in $y = e$ subgraphs, the worker will turn at least part of the even subgraph into a worker subgraph. The private and social returns of the new worker in both cases are equal to zero. The same happens, if the entrant contacts only firms in $y = \{w, e\}$ subgraphs. To sum up, private and social returns are only aligned, if workers receive the full surplus in even subgraph, and the social returns of the entering worker are above the private returns if firms receive the full surplus in even subgraphs.

Consider now the social and private return of a new vacancy. Denote the social return by S^q, where the superscript q denotes the set of possible subgraphs where workers that redirected their applications to the new vacancy would have ended up in case the vacancy did not enter. The set of subgraphs is given by $q \in \{f, w, e, \{f, w\}, \{f, e\}, \{w, e\}, \{f, w, e\}\}$. The private return is similarly denoted by P_k^q.

Table 3 summarizes all possible cases.

Table 3: Social and private returns of a new vacancy

new vacancy receives applications from workers in x subgraphs	social return S^x	private return (wages) P_f^x	P_w^x
$x = f$	0	0	0
$x = w$	$(0, 1)$	1	$(0, 1)$
$x = e$	0	0	0
$x = \{f, w\}$	$(0, 1)$	1	$(0, 1)$
$x = \{f, e\}$	0	0	0
$x = \{w, e\}$	$(0, 1)$	1	$(0, 1)$
$x = \{f, w, e\}$	$(0, 1)$	1	$(0, 1)$

Note first, that an additional match is only possible, if the new vacancy receives at least one application from a worker who in the absence of the new vacancy would have ended up in a worker subgraph, i.e., $q \in \{w, \{f, w\}, \{w, e\}, \{f, w, e\}\}$. The reason is simply that in all other cases (i.e., if workers were only part of a firm or even subgraph in the absence of the new vacancy), the workers that redirect one of their applications would have matched anyway. The social return S^q with $q \in \{w, \{f, w\}, \{w, e\}, \{f, w, e\}\}$ is smaller than one because of a *business-stealing effect*, i.e., the new vacancy attracts applications that would have otherwise gone to other firms. Some of those other firms do not match now but would have matched in the absence of the new firm. If this occurs, the social return equals zero. The private returns P_k^q also depend on whether firms ($k = f$) or workers ($k = w$) receive the full surplus in even subgraphs. If $k = w$ the new vacancy only benefits, i.e., has to pay a wage equal to zero, if it becomes part of a worker subgraph.

This happens, if $q \in \{w, \{f, w\}, \{w, e\}, \{f, w, e\}\}$ and if at least one of the respective workers remains in a worker subgraph. The probability that the worker remains in a worker subgraph need not be one, since there exists a positive probability that the new vacancy turns (part of) the worker subgraph into an even subgraph (e.g. if the worker subgraph has exactly one excess worker). This also implies that the private return of the new vacancy, P_k^q, is only equal to one if $q \in \{w, \{f, w\}, \{w, e\}, \{f, w, e\}\}$ and if firms receive the full surplus in even subgraphs, i.e., $k = f$.

If the new vacancy receives applications only from workers in $q = f$ subgraphs the new vacancy will become part of the firm subgraph and will have to pay a wage equal to one. If the new vacancy receives applications only from $q = e$ subgraphs the new vacancy will turn at least part of the even subgraph into a firm subgraph. Both the private and social return of the entrant equals zero in this case. The same happens, if the new vacancy receives applications from $q = \{f, e\}$ subgraphs.

In summary the content of Table 3 implies, if firms receive the full surplus in even subgraphs, i.e., $k = f$, private returns exceed social returns, and if $k = w$ and $\sum_q S^q = \sum_q P_w^q$, private and social returns are aligned.

The fact that the efficiency conditions for vacancy creation (25) and worker participation (29) are identical implies that vacancy creation is efficient, if worker participation is efficient. From the summary of Table 2 we know that worker participation is efficient, if workers receive the full surplus in even subgraphs, i.e., if $k = w$. This implies that if workers receive the full surplus in even subgraps (in expecttaion they receive the worker-optimal point in the core) i.e. $\sum_q S^q = \sum_q P_w^q$ if $k = w$. ■

Surprisingly, the negative business stealing externality and the positive externality that a new vacancy creates if it turns a worker graph into an even graph exactly offset each other if $k = w$ and this is what makes entry efficient. Again this extends the Kim and Kircher (2012) efficiency result to general a.

The proof of Proposition 5 implies as a Corollary that the expected wage in the decentralized economy where workers receive the full surplus in even subgraphs is equal to the expected wage that requires entry efficiency.

Corollary 2 *If workers receive the full surplus in even subgraphs, the expected wage in* *the decentralized economy is equal to,*

$$E[w] = \frac{u\left(1-\left(1-e^{-z(t^*)}\right)^a\right)}{u\left(1-\left(1-e^{-z(t^*)}\right)^a\right)+v\left(1-e^{-z(t^*)}-z(t^*)\,e^{-z(t^*)}\right)},$$

where $z(t^*)$ is given by equation (9).

7 Final remarks

When workers send applications to vacancies they create a bipartite network. Coordination frictions arise if workers and firms only observe their own links. We show that those frictions and the wage mechanism are in general not independent. Only wage mechanisms that allow for ex post competition generate the maximum matching on a realized network. We show that random search with ex post competition in wages leads to the maximum number of matches and is socially efficient in terms of vacancy creation, worker participation and the number of applications send out, if workers and not firms have the power to make offers. If firms are the ones that make offers, the resulting equilibrium is still efficient in terms of network formation and the number of matches on a given network but *not* efficient in terms of entr and search intensity.

References

[1] Albrecht, J., P.A. Gautier, S. Tan and S. Vroman, (2004), Matching with multiple applications, *Economic Letters*, vol. 84(3), pp. 311-314.

[2] Albrecht, J., P.A. Gautier and S. Vroman, (2006), Equilibrium directed search with multiple applications, *Review of Economic Studies*, vol. 73(4), pp. 869-891.

[3] Albrecht, J., P.A. Gautier and S. Vroman, (2013), Competing mechanisms and efficient entry, mimeo Vrije Universiteit Amsterdam.

[4] Boorman, S. (1975), A combinatorial optimization model for transmission of job information through contact networks, *Bell Journal of Economics*, vol. 6, 216-249.

[5] Burdett, K. and K.L. Judd, (1983), Equilibrium price dispersion, *Econometrica*, vol. 51(4), pp. 955-969.

[6] Burdett, K., S. Shi, and R. Wright, (2001), Pricing and matching with frictions, *Journal of Political Economy*, vol. 109(5), pp. 1060-1085.

[7] Calvó-Armengol, A. and M.O. Jackson, (2004), The effects of social networks on employment and inequality, *American Economic Review*, vol. 94(3), 426-454.

[8] Calvó-Armengol, A. and Y. Zenou, (2004), Job matching, social network and word-of-mouth communication, *Journal of Urban Economics*, vol. 57, 500-522.

[9] Chade, H. and L. Smith, (2006), Simultaneous search, *Econometrica*, vol. 74(5), pp. 1293-1307.

[10] Coles, M. and J. Eeckhout, (2003), "Indeterminacy and Directed Search", *Journal of Economic Theory*, vol. 111, pp. 265-276.

[11] Corominas-Bosch, M., (2004), Bargaining in a network of buyers and sellers, *Journal of Economic Theory*, vol. 115, pp. 35-77.

[12] Devroye, L., and P., Morin, (2003), Cuckoo hashing: further analysis, Inf Process Lett 86, pp. 215-219.

[13] Diamond, P.A. (1982), Aggregate demand management in search equilibrium, *Journal of Political Economy*, 90, 881-94.

[14] Eeckhout, J. and P. Kircher, (2010), Sorting vs Screening – Search Frictions and Competing Mechanisms, *Journal of Economic Theory*, vol. 145, pp. 1354-1385.

[15] Elliott, M., (2011). Search with multilateral bargaining, mimeo, Stanford University.

[16] Erdös, P. and A. Rényi, (1960), On the evolution of random graphs, *Publication of the mathematical institute of the Hungarian academy of sciences*, vol. 5, 17-61.

[17] Frieze, A. and P. Melsted, (2012), Maximum Matchings in Random Bipartite Graphs and the Space Utilization of Cuckoo Hash Tables, *Random Structures and Algorithms*, vol. 41(3), 334-364.

[18] Fontaine, F., (2004), Why are similar workers paid differently? The role of social networks, IZA discussion paper 1786, Bonn.

[19] Galenianos, M. and P. Kircher, (2009), Directed search with multiple job applications, *Journal of Economic Theory*, vol. 114(2), pp. 445-471.

[20] Galeotti, A., (2010), Strategic Information Transmission in Networks, mimeo, University of Essex.

[21] Gautier, P.A. and J.L. Moraga-Gonzalez, (2004), *Strategic wage setting and random search with multiple applications*, Tinbergen Institute discussion paper 04-063/1, Tinbergen Institute.

[22] Gautier, P.A. and R. Wolthoff, (2009), Simultaneous search with heterogeneous firms and ex-post competition, *labor Economics*, vol. 16(3), 311-19.

[23] Ioannides, Y.M., (2004), Random Graphs and Social Networks: An Economics Perspective, mimeo Tufts University.

[24] Julien, B., J. Kennes and I. King, (2000), Bidding for labor, *Review of Economic Dynamics*, vol. 3(4), pp. 619-649.

[25] Julien, B., J. Kennes and I. King, (2006), The Mortensen rule and efficient coordination unemployment, *Economics Letters*, vol. 90(2), 149-155.

[26] Kim, K. and P. Kircher, (2012), Efficient Cheap Talk in Directed Search: On the Non-essentiality of Commitment in Market Games, mimeo University of Edinburg.

[27] Kircher, P., (2009), Efficiency of simultaneous search, *Journal of Political Economy*, vol. 117(5), pp. 861- 913.

[28] Kranton, R.E. and D. F. Minehart, (2000), Competition for goods in buyer-seller networks, *Review of Economic Design*, vol. 5, pp. 301-331.

[29] Kranton, R.E. and D. F. Minehart, (2001), A theory of buyer-seller networks, *American Economic Review*, vol. 91(3), 485-508.

[30] Kutzelnigg, R., (2006), Bipartite random graphs and Cuckoo Hashing, *Proceedings of the fourth colloquium on mathematics and computer science.*

[31] Manea, M., (2011), Bargaining in stationary networks, *American Economic Review* 101, 2042-2080.

[32] McAfee (1993), Mechanism design by competing sellers, *Econometrica* 61-6, 1281-1312.

[33] Moen, E., (1997), Competitive search equilibrium, *Journal of Political Economy*, vol. 105(2), pp. 385-411.

[34] Mortensen, D., (1982), Property rights and efficiency of mating, racing, and related games. *American Economic Review* 72 (5), pp. 968–79.

[35] Motwani, R., R. Panigrahy and Y. Xu, (2006), Fractional Matching Via Balls-and-Bins, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques,* Lecture Notes in Computer Science, pp. 487-498.

[36] Pagh, R., and F. Rodler, (2004), Cuckoo Hashing, *Journal of Algorithms* 51(2), pp. 122-144.

[37] Peters M. and Severinov, (1997), Competition among sellers who offer auctions instead of prices, *Journal of Economic Theory*, vol.75, pp. 141-179.

[38] Pissarides, C.A., (2000), *Equilibrium unemployment theory*, 2nd edition, MIT Press, Cambridge.

[39] Shi, S., (2002), A directed search model of inequality with heterogeneous skills and skill-biased technology, *Review of Economic Studies*, vol. 69, pp. 467-491.

[40] Shimer, R., (2005), The assignment of workers to jobs in an economy with coordination frictions, *Journal of Political Economy*, vol. 113(5), pp. 996-1025.

8 Appendix

A Proofs

A.1 Proof of Lemma 1

Consider part (i). We prove the engagement result of (i) by contradiction. Denote the highest number of applicants that a firm in a worker subgraph has by $N^w \leq u$. Note, that all applicants of firms in a worker subgraph are in the same subgraph, since workers in a worker subgraph cannot be linked to a firm in an even or firm subgraph. Suppose firm j is part of a worker subgraph and it is not engaged in $t = N^w$. In $t = N^w$, according to strategy C2a, firm j must have offered the job to all its applicants and must have been rejected by all its applicants. Thus, all workers that are linked to firm j must be engaged with some other firm in the same worker subgraph, since workers always keep their best offer w^h according to strategy A2 and since workers in a worker subgraph are only linked to firms in the same worker subgraph. This leads to the desired contradiction, since the number of engaged workers cannot exceed the number of engaged firms in a worker subgraph.

Now consider the wage result of (i). In round $t = 1$ all firms start with the lowest possible offer, i.e., $w = 0$. According to the engagement result of part (i) all firms in a worker subgraph are engaged in round $t = u < T$. The counter offer strategy B2 rules out that an engaged firm will offer a wage $w \geq \Delta$ if the wage offered by the competing firm is no higher than $w = 0$. Thus, if we can rule out that a rejected firm in a worker subgraph offers $w \geq \Delta$ at any $t < T$, we have proven that the engaged workers of firms in worker subgraphs hold an offer no higher than $w = 0$ in any round $t < T$. The strategy C2b implies that a rejected firm only offers $w \geq \Delta$ in a round $t < T$ if $k = 0$. Since there are more workers than firms in a worker subgraph and since all workers are collectively linked to all firms in the subgraph, there is always at least one applicant without an offer, i.e., $k > 0$. Thus, the wage offers in a worker subgraph implied by the above strategies are no higher than $w = 0$ in any round $t < T$.

Consider now part (ii). We use a contradiction argument to rule out that firms in even subgraphs offer a wage $w = 2\Delta$. According to strategy C2b, a rejected firm only offers $w = 2\Delta$ if all its applicants are engaged and hold an offer $w = \Delta$. If all applicants of a rejected firm in an even subgraph are engaged, it must be the case that at least one of the applicants is engaged with a firm outside the even subgraph, because a firm in an even subgraph cannot be rejected, if all workers in even subgraphs are engaged with firms in even subgraphs. The Decomposition Theorem of Corominas-Bosch (2004) implies that workers in an even subgraph are either linked to firms in an even or to firms in a worker subgraphs. Thus, the applicant that is engaged with a firm outside the even subgraph must be engaged with a firm in a worker subgraph. This leads to the desired contradiction, since part (i) of the Lemma implies that the wage offers made by firms in worker subgraphs cannot be higher than $w = 0$. Thus, the rejected firm will according to strategy C2b never offer a wage $w = 2\Delta$ or higher. This implies that wages paid in even subgraphs are no higher than $w = \Delta$. According to strategy A2, since workers keep their best offers, it follows that any firm that offers $w = \Delta$ must be engaged. Since all firms start in round $t = 1$ with the lowest possible offer, i.e., $w = 0$, and since there are at most u workers linked to firms in even subgraphs, it takes at most $t = 2u$ rounds of rejections (where the wage offers $w = 0$ are rejected) until a firm offers for the first time the wage $w = \Delta$ and becomes engaged.

Now consider the engagement result of (iii). Denote the number of workers in a firm subgraph by $u^f \leq u$. Note, that all v^f firms in the respective firm subgraph are only linked to their respective u^f applicants in the same firm subgraph. Thus, since there are more firms than workers in a firm subgraph, i.e., $v^f > u^f$, at least one firm must always be rejected in any round. Strategy C2a implies that the rejected firms first choose one of the k applicants without an offer (if present) and offer her a wage $w = 0$. Thus, it takes at most $t = u^f$ rounds until all workers in a firm subgraph are engaged.

Now consider the wage result of (iii). The engagement result of part (iii) implies that all workers are engaged in round $t = u^f$, i.e., $k = 0$, and at least one firm is rejected. The rejected firm will according to strategy C2b choose one of the applicants with the lowest best offer, i.e., $\underline{w}^h = \min W^N$, and offer this applicant one cent more, i.e., $w = \underline{w}^h + \Delta$ if $\underline{w}^h < 1$. If the offer $w = \underline{w}^h + \Delta$ does not attract a worker, i.e., the rejected firm does not become engaged (which can happen, if the already engaged firm matches the offer according to strategy B2), the firm can offer $w = \underline{w}^h + \Delta$ to other (potential) applicants that hold an offer $\underline{w}^h$. After at most u^f rounds all applicants will hold an offer $\underline{w}^h + \Delta$ and the firm will either be engaged or still remain rejected. Since there is at least one rejected firm each round, wages will always increase by Δ after at most u^f rounds, i.e., after all workers experienced wage increases by Δ. By induction firms will increase their offers according to strategy C2b up to $w = 1$. Thus, there exists a round $t \leq u/\Delta$, in which all workers in a firm subgraph hold an offer $w^h = 1$, which they accept. ■

A.2 Proof of Lemma 2

The fact that wages differ across subgraphs enables firms to update their belief on whether they are in a firm subgraph or not. If firms have no applicant, i.e., $N = 0$, they are part of a firm subgraph by definition, i.e., $b_{j,0}(0, .) = 1$. Firms with at least one applicant, i.e., $N > 0$, start with a belief $b_{j,0}\left(N, W^N\right) \in (0, 1)$ that is equal to the ex-ante probability to be in a firm subgraph given N.

Consider part (i): Lemma 1 implies that only firms in firm and even subgraphs observe wage offers $w^h \geq \Delta$ in rounds $t \geq 2u$. Furthermore, at least one firm in each firm subgraph that observes $w^h \geq \Delta$ is rejected. These rejected firms form the belief $b_{j,t}\left(N, \left\{W^{N-m-n}, \ \Delta^m, 1^n\right\}\right) = 1$ in any round $t \geq 2u$, because parts (i) and (ii) of Lemma 1 imply that they would not have been rejected if they were in an worker- or even subgraph.

Consider part (ii): In round $t = u/\Delta$ all workers in firm subgraphs will have accepted a wage offer $w = 1$. Thus, at $t = u/\Delta$ all firms that are not in firm subgraphs, i.e., observe $W^N = \left\{\emptyset^k, 0^{N-k-m}, \Delta^m\right\}$, can infer that they are either in a worker- or even subgraph, i.e., $b_{j,T}\left(N,\left\{\varnothing^k,0^{N-k-m},\Delta^m\right\}\right)=0.$ ■

For other values of $W^N\{\cdot\}$ beliefs can be between 0 and 1. However, actions only depend on beliefs in the final round so we do not care about them.

A.3 Proof of Proposition 1

Consider first the strategies in round T.

Clearly, the workers' strategy A1 to accept the best offer in $t=T$ maximizes the workers' payoff.

Also, the engaged firms' strategy B1 of matching the outside offer in round $t=T$ is profit maximizing, since engaged firms that do not match outside offers would remain idle and earn a profit of zero.

Let us now turn to the strategies C1a and C1b of rejected firms. If there are some applicants without an offer, i.e., $k>0$, and if the belief that there are other competing firms is equal to zero, i.e., $b_{j,T}\left(N,\left\{\varnothing^k,0^{N-k-m},\Delta^m\right\}\right)=0$ at $t=T$, then the action implied by strategy C1a, i.e., the firm should offer a wage $w=0$ to one of the applicants without an offer, is profit maximizing. This follows from $b_{j,T}\left(N,\left\{\varnothing^k,0^{N-k-m},\Delta^m\right\}\right)=0$, i.e., from the fact that the rejected firm believes that all other firms in the same worker or even subgraph are according to Lemma 1 engaged with other workers and will therefore not make an offer to one of the k applicants that does not hold an offer. If the firm observes a set of wage offers that differs from the ones stated in Lemma 1 and has a belief $b_{j,t}\left(N,W^N\right)\in(0,1)$ it is optimal to follow the action implied by the second part of strategy C1a as characterized in Gautier and Moraga-Gonzalez (2004). Finally, if a rejected firm observes that all its applicants hold an offer in round T, the action implied by strategy C1b, i.e., offering $w=1$ to one of the candidates, is equally profitable as any other action, since the engaged firms will match outside offers (as implied by strategy B1). Thus, the rejected firm cannot do better by deviating from strategy C1b.

Consider now the strategies in any round $t<T$.

The workers' strategy, A2 to keep the best offer w^h is a dominant strategy, because the rejected firms' strategies C2a and C2b imply that keeping a lower offer can lead to a lower payoff for the worker without increasing the chances of receiving better offers in the future.

Consider now strategy B2 for engaged firms. Obviously, a firm can only be engaged, if it offered a wage $w^h\geq 0$ in the past. Due to the tie-breaking rule, which implies that workers prefer their engaged firm over an outside firm in case both firms offer the same wage, matching an outside offer $w^h=0$ is optimal since it ensures that the firm stays engaged at zero cost. Now consider the different cases if the outside offer satisfies $w^h\geq\Delta$. Denote the highest wage offer that the engaged worker (A) holds by w^h and the highest offer that another applicant holds by $\widetilde{w}^h$. If all other applicants hold the same or a higher offer, i.e., $\widetilde{w}^h\geq w^h$, it is a dominant strategy to match the outside offer of worker A, since it is the least costly way for the firm to stay engaged. It is also optimal to match the outside offer of worker A if one of the other applicants (B) holds an offer $\widetilde{w}^h=w^h-\Delta$, because otherwise the engaged firm must offer applicant B $\widetilde{w}^h+\Delta=w^h$ to have the chance to become engaged. Note, that in the later case the firm cannot be sure that it will become engaged (since other firms might also compete for the same worker). If one of the other applicants (B) holds no offer, matching the outside offer of worker A $w^h \geq \Delta$ cannot be optimal. To see this consider the different subgraphs a firm can be in. If the firm is in a worker subgraph, offering the job to applicant B generates profit 1 while matching the outside offer of worker A generates $1 - \Delta$ or less. If the firm is part of an even subgraph, offering the job to applicant B leads to the expected profit $\gamma + (1 - \gamma)(1 - \Delta)$, where $\gamma > 0$ equals the probability that the firm will pay the wage $w = 0$, while matching the outside offer of worker A generates $1 - \Delta$ for sure. If the firm is part of a firm subgraph, profits are driven down to zero and the firm may as well not match the outside offer of applicant A and offer the job to applicant B. Thus, not matching an outside offer $w^h \geq \Delta$, if one of the other applicants holds no offer is weakly dominating. The same is true, if applicant B holds an offer $\widetilde{w}^h < w^h - \Delta$, where $\widetilde{w}^h \geq 0$. To see this, note first that $\widetilde{w}^h \geq 0$ and $\widetilde{w}^h < w^h - \Delta$ imply $w^h > \Delta$. According to strategy C2b a firm (1) offers $w^h > \Delta$ only if $W^N = \left\{ W^{N-k-l-m-n}, \text{Ø}^k, 0^l, \Delta^m, 1^n \right\}$ with $k = l = 0$ and $m > 0$. Lemmas 1 and 2 then imply that firm 1 that offered w^h to worker worker A is part of a firm subgraph. Since a firm in a firm subgraph will eventually pay a wage $w = 1$, it is optimal for the engaged firm not to compete with firm 1 in the firm subgraph, i.e., not to match w^h, but to offer the job to applicant B at the wage $\widetilde{w}^h + \Delta < w^h$.

The strategies C2a and C2b of rejected firms to pick (one of) the applicant(s) with the lowest offer and to offer this applicant the job at the lowest possible wage are also optimal. Any deviation would lead to lower profits. To see this consider deviations depending on the set of wage offers W^N and the type of subgraph the firm is in. Suppose at least one applicant holds no offer, i.e., $W^N = \left\{ W^{N-k-l-m-q}, \text{Ø}^k, 0^l, \Delta^m, 1^q \right\}$ with $k > 0$, and firm (1) chooses in contrast to strategy C2a to offer the job to some engaged worker (A) that holds an offer $w^h \geq 0$. The profit of this deviating strategy will be $1 - \Delta$ in case the worker is part of a worker subgraph, since firm 1 has to offer a wage $w = \Delta$ in order to become engaged. However, playing strategy C2a and offering the job to an applicant without an offer ensures according to Lemma 1 a profit of 1. A similar argument implies that a deviation leads to an expected profit of $\gamma + (1 - \gamma)(1 - \Delta)$ in case firm 1 is part of an even subgraph. If firm 1 is part of a firm subgraph deviating is equally profitable as playing strategy C2a. Thus, for a firm with belief $b_{j,t}\left(N, W^N\right) \in (0, 1)$ action C2a maximizes expected profits. Next, suppose that all applicants hold an offer and at least one applicant holds an offer $w^h = 0$, i.e., $W^N = \left\{ W^{N-k-l-m-q}, \text{Ø}^k, 0^l, \Delta^m, 1^q \right\}$ with $k = 0$ and $l > 0$, and the deviating firm chooses in contrast to strategy C2b to offer the job to some engaged worker (A) that holds an offer $w^h \geq \Delta$. Note that Lemma 1 implies that wage offers $w^h = \Delta$ are only observed, if the worker is part of an even or firm subgraph. The profit of this deviation will be $1 - 2\Delta$ in case the worker is part of an even subgraph, since the deviating firm has to offer a wage $w = 2\Delta$ in order to become engaged. Following strategy C2b and offering the job to an applicant with an offer $w^h = 0$ ensures a profit $1 - \Delta$ since wages in an even subgraph are no higher than Δ (see Lemma 1). If the deviating firm is part of a firm subgraph deviating is equally profitable as playing strategy C2b, since profits are equal to zero anyway. Thus, without knowing the subgraph deviating is never profitable. Finally, suppose all applicants hold an offer $w^h \geq \Delta$, i.e., $W^N = \left\{ W^{N-k-l-m-q}, \text{Ø}^k, 0^l, \Delta^m, 1^q \right\}$ with $k = l = 0$ and $m \geq 0$, and the deviating firm chooses in contrast to strategy C2b to offer the job to some engaged worker that holds an offer $w^h > \underline{w}^h = \min W^N$. Note that Lemmas 1 and 2 imply that wage offers $w^h \geq \Delta$ are only observed by rejected firms, if they are part of a firm subgraph. Thus, offering the wage $w = \underline{w}^h + \Delta$ (as implied by strategy C2b) or any other wage $w \in \left[\underline{w}^h + \Delta, 1\right]$ generates the same profit, as the wage in a firm subgraph will eventually increase up to $w = 1$. To sum up, deviating from strategy C2b without knowing the subgraph yields a strictly lower expected payoff. ■

A.4 Proof of Lemma 5

Differentiating the RHS of equation (8) – dropping the time index for simplicity – implies,

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{au - v^1}{v}\right) &= \frac{1}{v}\left(\left(-a - \frac{dv^1}{dt}\right) - \frac{dv}{dt}\left(\frac{au - v^1}{v}\right)\right), \\
&= \frac{1}{v}\left(-a + 1 + \frac{a-1}{au}v^1 - \frac{a-1}{au}\frac{z^2 e^{-z}}{1 - e^{-z} - ze^{-z}}v\right) \\
&+ \frac{a-1}{au}\frac{z^2 e^{-z}}{1 - e^{-z} - ze^{-z}}\frac{z\left(1 - e^{-z}\right)}{1 - e^{-z} - ze^{-z}}, \\
&= -\left(\frac{a-1}{au}\frac{z\left(1 - e^{-z}\right)}{1 - e^{-z} - ze^{-z}} + \frac{a-1}{au}\frac{z^2 e^{-z}}{1 - e^{-z} - ze^{-z}}\right) \\
&+ \frac{a-1}{au}\frac{z^2 e^{-z}}{1 - e^{-z} - ze^{-z}}\frac{z\left(1 - e^{-z}\right)}{1 - e^{-z} - ze^{-z}}, \\
&= -\frac{a-1}{au}\frac{z\left(1 - e^{-z} + ze^{-z}\right)\left(1 - e^{-z} - ze^{-z}\right) - z^3 e^{-z}\left(1 - e^{-z}\right)}{\left(1 - e^{-z} - ze^{-z}\right)^2}, \\
&= -\frac{a-1}{au}z\frac{\left(1 - e^{-z}\right)^2 - z^2 e^{-z}}{\left(1 - e^{-z} - ze^{-z}\right)^2}.
\end{aligned}
$$

Differentiating the LHS of equation (8) implies

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{z\left(1 - e^{-z}\right)}{1 - e^{-z} - ze^{-z}}\right) &= \frac{\left(1 - e^{-z} + ze^{-z}\right)\left(1 - e^{-z} - ze^{-z}\right) - z^2 e^{-z}\left(1 - e^{-z}\right)}{\left(1 - e^{-z} - ze^{-z}\right)^2}\frac{dz}{dt}, \\
&= \frac{\left(1 - e^{-z}\right)^2 - z^2 e^{-z}}{\left(1 - e^{-z} - ze^{-z}\right)^2}\frac{dz}{dt}.
\end{aligned}
$$

Equating RHS and LHS implies,

$$\frac{1}{z}\frac{dz}{dt} = -\frac{a-1}{au}$$

Integrating implies

$$\frac{z^a}{u^{a-1}} = C,$$

Using the starting conditions $z\left(0\right)$ and $u\left(0\right) = u$ and the function $u\left(t\right) = u - t$ implies equations (4) and (5).

Going back to $dv/dt = (dv/dz)(dz/dt)$, implies,

$$-\frac{a-1}{au}\frac{z^2e^{-z}}{\left(1-e^{-z}-ze^{-z}\right)^2}v = -\frac{a-1}{au}z\frac{dv}{dz},$$
$$\text{or } \frac{dv}{dz} = \frac{ze^{-z}}{\left(1-e^{-z}-ze^{-z}\right)^2}v.$$

Integrating gives equation (6) given that $v(0) = v\left(1-e^{-z(0)}-z(0)\,e^{-z(0)}\right)$.

We now solve for v^1 by substituting v and u into equation (8). Rearranging then implies equation (7). ■

A.5 Proof of Lemma 8

The respective differential equations characterize the evolution of vacancies with one and with at least two applications in subgroup $c \in C$, i.e.,

$$\frac{dv_c^1(t)}{dt} = -\kappa_c(t) - \frac{p_c a - \kappa_c(t)}{p_c a u(t)}v_c^1(t) + \frac{p_c a - \kappa_c(t)}{p_c a u(t)}\frac{z_c(t)^2 e^{-z_c(t)}}{1-e^{-z_c(t)}-z_c(t)\,e^{-z_c(t)}}v_c(t),$$
$$\frac{dv_c(t)}{dt} = -\frac{p_c a - \kappa_c(t)}{p_c a u(t)}\frac{z_c(t)^2 e^{-z_c(t)}}{1-e^{-z_c(t)}-z_c(t)\,e^{-z_c(t)}}v_c(t).$$

and $u(t) = U - t$ and where $z_c(t)$ satisfy,

$$\frac{p_c a u(t) - v_c^1(t)}{v_c(t)} = \frac{z_c(t)\left(1-e^{-z_c}\right)}{1-e^{-z_c(t)}-z_c(t)\,e^{-z_c(t)}} \tag{31}$$

and the Boundary conditions are given by $u(0) = u$, $v_c^1(0) = v_c z_c(0)\,e^{-z_c(0)}$, $v_c(0) = v_c\left(1-e^{-z_c(0)}-z_c(0)\,e^{-z_c(0)}\right)$, and $z_c(0) = p_c a z/v_c$ with $\sum_{c\in C} v_c = v$ and $\sum_{c\in C} p_c = 1$.

Differentiating the RHS of equation (31) implies (following the steps in the Proof of Lemma 5),

$$\frac{d}{dt}\left(\frac{p_c a u(t) - v_c^1(t)}{v_c(t)}\right) = -\frac{p_c a - \kappa_c(t)}{p_c a u(t)}z_c(t)\frac{\left(1-e^{-z_c(t)}\right)^2 - z_c(t)^2 e^{-z_c(t)}}{\left(1-e^{-z_c(t)}-z_c(t)\,e^{-z_c(t)}\right)^2}$$

Differentiating the LHS of equation (31) implies

$$\frac{d}{dt}\left(\frac{z_c(t)\left(1-e^{-z_c}\right)}{1-e^{-z_c(t)}-z_c(t)\,e^{-z_c(t)}}\right) = \frac{\left(1-e^{-z_c(t)}\right)^2 - z_c(t)^2 e^{-z_c(t)}}{\left(1-e^{-z_c(t)}-z_c(t)\,e^{-z_c(t)}\right)^2}\frac{dz_c(t)}{dt}.$$

Equating RHS and LHS implies,

$$\frac{p_c}{z_c(t)}\frac{dz_c(t)}{dt} = -\frac{p_c a - \kappa_c(t)}{au(t)}.$$

Taking the sum over all subgroups $c \in C$ implies,

$$\sum_{c\in C}\frac{p_c}{z_c(t)}\frac{dz_c(t)}{dt} = -\frac{a-1}{au(t)},$$

since $\sum_{c\in C}\kappa_c(t)=\sum_{c\in C}p_c=1$.

Integrating[16] implies

$$\frac{\prod_{c\in C} z_c(t)^{ap_c}}{u(t)^{a-1}}=C,$$

Using the starting conditions $z_c(0)$ and $u(0)=u$ and the function $u(t)=u-t$ implies equations (17) and (18).

Going back to $dv_c(t)/dt=(dv_c(t)/dz_c(t))(dz_c(t)/dt)$, implies,

$$-\frac{p_c a-\kappa_c(t)}{p_c a u(t)}\frac{z_c(t)^2 e^{-z_c(t)}}{1-e^{-z_c(t)}-z_c(t)e^{-z_c(t)}}v_c(t)=-\frac{p_c a-\kappa_c(t)}{au(t)}\frac{z_c(t)}{p_c}\frac{dv_c(t)}{dz_c(t)}$$

$$\text{or }\frac{dv_c(t)}{dz_c(t)}=\frac{z_c(t)e^{-z_c(t)}}{1-e^{-z_c(t)}-z_c(t)e^{-z_c(t)}}v_c(t).$$

Integrating gives equation (19) given that $v_c(0)=v_c\left(1-e^{-z_c(0)}-z_c(0)e^{-z_c(0)}\right)$.

We now solve for $v_c^1(t)$ by substituting $v_c(t)$ and $u(t)$ into equation (31). Rearranging then implies equation (20). ■

A.6 Proof of Proposition 3

We start by showing that random search is optimal in case of two subgroup of vacancies r and b. The general result then follows by induction. Denote $\mu(G)=t^*+\min\left[u(t^*),\sum_{c\in\{r,b\}}v_c(t^*)\right]$. Using the results of Lemma 8 we get

$$\mu(G)=u-u\left(\left(\frac{z_r(t^*)}{z_r(0)}\right)^{p_r}\left(\frac{z_b(t^*)}{z_b(0)}\right)^{p_b}\right)^{\frac{a}{a-1}}+\sum_{c\in\{r,b\}}v_c\left(1-e^{-z_c(t^*)}-z_c(t^*)e^{-z_c(t^*)}\right), \tag{32}$$

where $v_r=v-v_b$ and $p_r=1-p_b$.

[16] The solution can be verified using the Implicit Function Theorem (on $0=\prod_{c\in C}z_c(t)^{p_c}-Cu(t)^{a-1}$),

$$\frac{dz_c(t)}{dt}=-\frac{\sum_{c\notin C}\left(\frac{ap_c}{z_c(t)}\prod_{c\in C}z_c(t)^{ap_c}\right)\frac{dz_c(t)}{dt}-C(a-1)u(t)^{a-2}\frac{du(t)}{dt}}{\frac{ap_c}{z_c(t)}\prod_{c\in C}z_c(t)^{ap_c}}.$$

The FOC's with respect to v_b and p_b are given by,

$$\begin{aligned}\frac{\partial \mu(G)}{\partial v_b} &= -u\frac{a}{a-1}\Psi\left(\frac{p_b}{v_b}-\frac{p_r}{v_r}\right) \\ &- u\frac{a}{a-1}\Psi\left(\frac{p_b}{z_b(t^*)}\left(\frac{dz_b}{dv_b}-\frac{dz_b}{dv_r}\right)-\frac{p_r}{z_r(t^*)}\left(\frac{dz_r}{dv_r}-\frac{dz_r}{dv_b}\right)\right) \\ &+ \left(1-e^{-z_b(t^*)}-z_b(t^*)e^{-z_b(t^*)}\right)-\left(1-e^{-z_r(t^*)}-z_r(t^*)e^{-z_r(t^*)}\right) \\ &+ v_b z_b(t^*)e^{-z_b(t^*)}\left(\frac{dz_b}{dv_b}-\frac{dz_b}{dv_r}\right)-v_r z_r(t^*)e^{-z_r(t^*)}\left(\frac{dz_r}{dv_r}-\frac{dz_r}{dv_b}\right) \\ &= 0,\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{\partial \mu(G)}{\partial p_b} &= -u\frac{a}{a-1}\Psi\ln\left(\frac{z_b(t^*)}{z_b(0)}\frac{z_r(0)}{z_r(t^*)}\right) \\ &- u\frac{a}{a-1}\Psi\left(\frac{p_b}{z_b(t^*)}\left(\frac{dz_b}{dp_b}-\frac{dz_b}{dp_r}\right)-\frac{p_r}{z_r(t^*)}\left(\frac{dz_r}{dp_r}-\frac{dz_r}{dp_b}\right)\right) \\ &+ v_b z_b(t^*)e^{-z_b(t^*)}\left(\frac{dz_b}{dp_b}-\frac{dz_b}{dp_r}\right)-v_r z_r(t^*)e^{-z_r(t^*)}\left(\frac{dz_r}{dp_r}-\frac{dz_r}{dp_b}\right) \\ &= 0,\end{aligned} \tag{34}$$

with

$$\begin{aligned}\Psi &= \left(\left(\frac{z_r(t^*)}{z_r(0)}\right)^{p_r}\left(\frac{z_b(t^*)}{z_b(0)}\right)^{p_b}\right)^{\frac{a}{a-1}} \text{ and,} \\ \frac{dz_c}{dv_k} &= \frac{dz_c(t^*)}{dv_k}, \frac{dz_c}{dp_k} = \frac{dz_c(t^*)}{dp_k} \text{ with } c,k\in\{r,b\}.\end{aligned}$$

According to the equation (20) the following two functions determine $z_r(t^*)$ and $z_b(t^*)$ simultaneously, i.e.,

$$R = \left(\left(\frac{z_r(t^*)}{z_r(0)}\right)^{p_r}\left(\frac{z_b(t^*)}{z_b(0)}\right)^{p_b}\right)^{\frac{a}{a-1}} - \frac{z_r(t^*)\left(1-e^{-z_r(t^*)}\right)}{z_r(0)} = 0, \tag{35}$$

$$B = \left(\left(\frac{z_r(t^*)}{z_r(0)}\right)^{p_r}\left(\frac{z_b(t^*)}{z_b(0)}\right)^{p_b}\right)^{\frac{a}{a-1}} - \frac{z_b(t^*)\left(1-e^{-z_b(t^*)}\right)}{z_b(0)} = 0. \tag{36}$$

Using the implicit function theorem allows us to determine the derivatives of $z_r(t^*)$ and

$z_b(t^*)$ with respect to v_b, v_r and p_b, p_r, i.e.,

$$\frac{dz_r}{dv_b} = \frac{B'_{v_b}R'_{z_b} - B'_{z_b}R'_{v_b}}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}}, \text{ and } \frac{dz_b}{dv_r} = \frac{B'_{z_r}R'_{v_r} - B'_{v_r}R'_{z_r}}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}}, \tag{37}$$

$$\frac{dz_b}{dv_b} = \frac{B'_{z_r}R'_{v_b} - B'_{v_b}R'_{z_r}}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}}, \text{ and } \frac{dz_r}{dv_r} = \frac{B'_{v_r}R'_{z_b} - B'_{z_b}R'_{v_r}}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}}, \tag{38}$$

$$\frac{dz_r}{dp_b} = \frac{B'_{p_b}R'_{z_b} - B'_{z_b}R'_{p_b}}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}}, \text{ and } \frac{dz_b}{dp_r} = \frac{B'_{z_r}R'_{p_r} - B'_{p_r}R'_{z_r}}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}}, \tag{39}$$

$$\frac{dz_b}{dp_b} = \frac{B'_{z_r}R'_{p_b} - B'_{p_b}R'_{z_r}}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}}, \text{ and } \frac{dz_r}{dp_r} = \frac{B'_{p_r}R'_{z_b} - B'_{z_b}R'_{p_r}}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}}. \tag{40}$$

Note, that $p_r = p_b = 1/2$ and $v_r = v_b = 1/2v$ guarantee that the first derivatives are equal to zero, since $p_r = p_b = 1/2$ and $v_r = v_b = 1/2v$ imply $z_r(0) = z_b(0)$, $z_r(t^*) = z_b(t^*)$, and

$$\frac{dz_r}{dv_b} = \frac{dz_b}{dv_r}, \frac{dz_b}{dv_b} = \frac{dz_r}{dv_r}, \frac{dz_r}{dp_b} = \frac{dz_b}{dp_r}, \frac{dz_b}{dp_b} = \frac{dz_r}{dp_r}. \tag{41}$$

To show that random search, i.e., $p_r = p_b = 1/2$ and $v_r = v_b = 1/2v$, constitutes a maximum, we need to show that the Hessian matrix is negative semidefinite. We first derived the second derivatives and then evaluated them at $p_r = p_b = 1/2$ and $v_r = v_b = 1/2v$. The results are as follows,

$$\begin{aligned}\frac{\partial^2\mu(G)}{\partial(v_b)^2} &= 2u\frac{a}{a-1}\frac{p_b}{(v_b)^2}\Psi \\ &+ 2\left(u\frac{a}{a-1}\frac{p_b}{z_b(t^*)^2}\Psi + v_b(1 - z_b(t^*))e^{-z_b(t^*)}\right)\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right)^2 \\ &- 2\left(u\frac{a}{a-1}\frac{p_b}{z_b(t^*)}\Psi - v_b z_b(t^*)e^{-z_b(t^*)}\right)\frac{\partial}{\partial v_b}\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right) \\ &+ 4z_b(t^*)e^{-z_b(t^*)}\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right),\end{aligned} \tag{42}$$

$$\begin{aligned}\frac{\partial^2\mu(G)}{\partial(p_b)^2} &= 2u\frac{a}{a-1}\frac{1}{p_b}\Psi \\ &- 4u\frac{a}{a-1}\frac{1}{z_b(t^*)}\Psi\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right) \\ &+ 2\left(u\frac{a}{a-1}\frac{p_b}{z_b(t^*)^2}\Psi + v_b(1 - z_b(t^*))e^{-z_b(t^*)}\right)\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right)^2 \\ &- 2\left(u\frac{a}{a-1}\frac{p_b}{z_b(t^*)}\Psi - v_b z_b(t^*)e^{-z_b(t^*)}\right)\frac{\partial}{\partial p_b}\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right),\end{aligned} \tag{43}$$

$$\frac{\partial^2 \mu(G)}{\partial v_b \partial p_b} = -2u\frac{a}{a-1}\Psi\frac{1}{v_b} \tag{44}$$
$$- 2u\frac{a}{a-1}\frac{1}{z_b(t^*)}\Psi\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right)$$
$$+ 2\left(u\frac{a}{a-1}\frac{p_b}{z_b(t^*)^2}\Psi + v_b(1 - z_b(t^*))e^{-z_b(t^*)}\right)\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right)\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right)$$
$$- 2\left(u\frac{a}{a-1}\frac{p_b}{z_b(t^*)}\Psi - v_b z_b(t^*)e^{-z_b(t^*)}\right)\frac{\partial}{\partial p_b}\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right)$$
$$+ 2z_b(t^*)e^{-z_b(t^*)}\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right),$$

$$\frac{\partial^2 \mu(G)}{\partial p_b \partial v_b} = -2u\frac{a}{a-1}\frac{1}{v_b}\Psi \tag{45}$$
$$- 2u\frac{a}{a-1}\frac{1}{z_b(t^*)}\Psi\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right)$$
$$+ 2\left(u\frac{a}{a-1}\frac{p_b}{z_b(t^*)^2}\Psi + v_b(1 - z_b(t^*))e^{-z_b(t^*)}\right)\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right)\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right)$$
$$- 2\left(u\frac{a}{a-1}\frac{p_b}{z_b(t^*)}\Psi - v_b z_b(t^*)e^{-z_b(t^*)}\right)\frac{\partial}{\partial v_b}\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right)$$
$$+ 2z_b(t^*)e^{-z_b(t^*)}\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right).$$

To determine the second derivatives we need to derive expressions for $\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right)$ and $\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right)$ as well as their derivatives. Using equations (37) to (40) we get,

$$\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r} = \frac{B'_{z_r}R'_{v_b} - B'_{v_b}R'_{z_r} - B'_{z_r}R'_{v_r} + B'_{v_r}R'_{z_r}}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}}, \tag{46}$$
$$\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r} = \frac{B'_{z_r}R'_{p_b} - B'_{p_b}R'_{z_r} - B'_{z_r}R'_{p_r} + B'_{p_r}R'_{z_r}}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}}. \tag{47}$$

Using equations (35) and (36) we can obtain the following expressions for the derivatives of R and B, i.e.,

$$R'_{v_b} = p_b\tfrac{a}{a-1}\tfrac{1}{v_b}\Psi, \qquad R'_{v_r} = p_r\tfrac{a}{a-1}\tfrac{1}{v_r}\Psi - \tfrac{1}{v_r}\frac{z_r(t^*)\left(1-e^{-z_r(t^*)}\right)}{z_r(0)},$$
$$B'_{v_r} = p_r\tfrac{a}{a-1}\tfrac{1}{v_r}\Psi, \qquad B'_{v_b} = p_b\tfrac{a}{a-1}\tfrac{1}{v_b}\Psi - \tfrac{1}{v_b}\frac{z_b(t^*)\left(1-e^{-z_b(t^*)}\right)}{z_b(0)},$$
$$R'_{p_b} = \tfrac{a}{a-1}\left(\ln\left(\tfrac{z_b(t^*)}{z_b(0)}\right) - 1\right)\Psi, \qquad R'_{p_r} = \tfrac{a}{a-1}\left(\ln\left(\tfrac{z_r(t^*)}{z_r(0)}\right) - 1\right)\Psi + \tfrac{1}{p_r}\frac{z_r(t^*)\left(1-e^{-z_r(t^*)}\right)}{z_r(0)},$$
$$B'_{p_r} = \tfrac{a}{a-1}\left(\ln\left(\tfrac{z_r(t^*)}{z_r(0)}\right) - 1\right)\Psi, \qquad B'_{p_b} = \tfrac{a}{a-1}\left(\ln\left(\tfrac{z_b(t^*)}{z_b(0)}\right) - 1\right)\Psi + \tfrac{1}{p_b}\frac{z_b(t^*)\left(1-e^{-z_b(t^*)}\right)}{z_b(0)},$$
$$R'_{z_b} = p_b\tfrac{a}{a-1}\tfrac{1}{z_b(t^*)}\Psi, \qquad R'_{z_r} = p_r\tfrac{a}{a-1}\tfrac{1}{z_r(t^*)}\Psi - \frac{1-e^{-z_r(t^*)}+z_r(t^*)e^{-z_r(t^*)}}{z_r(0)},$$
$$B'_{z_r} = p_r\tfrac{a}{a-1}\tfrac{1}{z_r(t^*)}\Psi, \qquad B'_{z_b} = p_b\tfrac{a}{a-1}\tfrac{1}{z_b(t^*)}\Psi - \frac{1-e^{-z_b(t^*)}+z_b(t^*)e^{-z_b(t^*)}}{z_b(0)}.$$

We therefore get the following expressions for,

$$
\begin{aligned}
&B'_{z_r}R'_{v_b} - B'_{v_b}R'_{z_r} - B'_{z_r}R'_{v_r} + B'_{v_r}R'_{z_r} \\
&= p_r \frac{a}{a-1} \frac{1}{z_r(t^*)} \Psi \left(\frac{1}{v_r} \frac{z_r(t^*)\left(1-e^{-z_r(t^*)}\right)}{z_r(0)} + \frac{1}{v_b} \frac{z_b(t^*)\left(1-e^{-z_b(t^*)}\right)}{z_b(0)} \right) \\
&+ \left(\frac{p_b}{v_b} - \frac{p_r}{v_r} \right) \frac{a}{a-1} \Psi \frac{1-e^{-z_r(t^*)} + z_r(t^*)\, e^{-z_r(t^*)}}{z_r(0)} \\
&- \frac{1}{v_b} \frac{z_b(t^*)\left(1-e^{-z_b(t^*)}\right)}{z_b(0)} \frac{1-e^{-z_r(t^*)} + z_r(t^*)\, e^{-z_r(t^*)}}{z_r(0)},
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
&B'_{z_r}R'_{p_b} - B'_{p_b}R'_{z_r} - B'_{z_r}R'_{p_r} + B'_{p_r}R'_{z_r} \\
&= -p_r \frac{a}{a-1} \frac{1}{z_r(t^*)} \Psi \left(\frac{1}{p_b} \frac{z_b(t^*)\left(1-e^{-z_b(t^*)}\right)}{z_b(0)} + \frac{1}{p_r} \frac{z_r(t^*)\left(1-e^{-z_r(t^*)}\right)}{z_r(0)} \right) \\
&+ \ln \left(\frac{z_b(t^*)}{z_b(0)} \frac{z_r(0)}{z_r(t^*)} \right) \frac{a}{a-1} \Psi \frac{1-e^{-z_r(t^*)} + z_r(t^*)\, e^{-z_r(t^*)}}{z_r(0)} \\
&+ \frac{1}{p_b} \frac{z_b(t^*)\left(1-e^{-z_b(t^*)}\right)}{z_b(0)} \frac{1-e^{-z_r(t^*)} + z_r(t^*)\, e^{-z_r(t^*)}}{z_r(0)},
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
&B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b} \\
&= p_r \frac{1-e^{-z_b(t^*)} + z_b(t^*)\, e^{-z_b(t^*)}}{z_b(0)} \left(\frac{z_r(t^*)\, e^{-z_r(t^*)}}{z_r(0)} - \frac{1}{a-1} \frac{1-e^{-z_r(t^*)}}{z_r(0)} \right) \\
&+ p_b \frac{1-e^{-z_r(t^*)} + z_r(t^*)\, e^{-z_r(t^*)}}{z_r(0)} \left(\frac{z_b(t^*)\, e^{-z_b(t^*)}}{z_b(0)} - \frac{1}{a-1} \frac{1-e^{-z_b(t^*)}}{z_b(0)} \right).
\end{aligned} \tag{50}
$$

Substituting back into equations (46) and (47) and applying the FOC, i.e., $p_b = p_r = 1/2$ and $v_b = v_r = 1/2v$, gives,

$$
\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r} = -\frac{1}{v_b} \frac{z_r(t^*)\left(1-e^{-z_r(t^*)}\right)}{1-e^{-z_r(t^*)} + z_r(t^*)\, e^{-z_r(t^*)}}, \tag{51}
$$

$$
\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r} = \frac{1}{p_b} \frac{z_r(t^*)\left(1-e^{-z_r(t^*)}\right)}{1-e^{-z_r(t^*)} + z_r(t^*)\, e^{-z_r(t^*)}}. \tag{52}
$$

To determine the derivatives of $\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right)$ and $\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right)$ let us first determine the derivatives of (48), (49) and (50). The results below show these derivatives evaluated at the FOC, i.e., $p_b = p_r = 1/2$ and $v_b = v_r = 1/2v$,

$$\frac{\partial}{\partial v_b}\left(B'_{z_r}R'_{v_b} - B'_{v_b}R'_{z_r} - B'_{z_r}R'_{v_r} + B'_{v_r}R'_{z_r}\right) \tag{53}$$
$$= -\left(\frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{z_r(t^*)} + (2 - z_r(t^*))\,e^{-z_r(t^*)}\right)\frac{1}{1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}}\frac{\Psi^2}{(v_b)^2}$$
$$-\frac{a}{a-1}\frac{1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}}{z_r(0)}\frac{1}{(v_b)^2}\Psi$$
$$+2\frac{1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}}{z_r(0)}\frac{1}{(v_b)^2}\Psi,$$

$$\frac{\partial}{\partial p_b}\left(B'_{z_r}R'_{p_b} - B'_{p_b}R'_{z_r} - B'_{z_r}R'_{p_r} + B'_{p_r}R'_{z_r}\right) \tag{54}$$
$$= -\left(\frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{z_r(t^*)} + (2 - z_r(t^*))\,e^{-z_r(t^*)}\right)\frac{1}{1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}}\frac{\Psi^2}{(p_b)^2}$$
$$-\frac{a}{a-1}\frac{1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}}{z_r(0)}\frac{1}{(p_b)^2}\Psi$$
$$+2\frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{z_r(0)}\frac{1}{(p_b)^2}\Psi,$$

$$\frac{\partial}{\partial p_b}\left(B'_{z_r}R'_{v_b} - B'_{v_b}R'_{z_r} - B'_{z_r}R'_{v_r} + B'_{v_r}R'_{z_r}\right) \tag{55}$$
$$= \left(\frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{z_r(t^*)} + (2 - z_r(t^*))\,e^{-z_r(t^*)}\right)\frac{1}{1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}}\frac{1}{p_b}\frac{1}{v_b}\Psi^2$$
$$+\frac{1}{a-1}\frac{1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}}{z_r(0)}\frac{1}{p_b}\frac{1}{v_b}\Psi$$
$$-\frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{z_r(0)}\frac{1}{p_b}\frac{1}{v_b}\Psi,$$

$$\frac{\partial}{\partial v_b}\left(B'_{z_r}R'_{p_b} - B'_{p_b}R'_{z_r} - B'_{z_r}R'_{p_r} + B'_{p_r}R'_{z_r}\right) \tag{56}$$
$$= \left(\frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{z_r(t^*)} + (2 - z_r(t^*))\,e^{-z_r(t^*)}\right)\frac{1}{1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}}\frac{1}{p_b}\frac{1}{v_b}\Psi^2$$
$$+\frac{1}{a-1}\frac{1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}}{z_r(0)}\frac{1}{p_b}\frac{1}{v_b}\Psi$$
$$-\frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{z_r(0)}\frac{1}{p_b}\frac{1}{v_b}\Psi,$$

$$\frac{\partial}{\partial v_b}\left(B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}\right) = \frac{\partial}{\partial p_b}\left(B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}\right) = 0. \tag{57}$$

The derivatives of $\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right)$ and $\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right)$ with respect to v_b and p_b, evaluated at the FOC, i.e., $p_b = p_r = 1/2$ and $v_b = v_r = 1/2v$, are therefore given by,

$$\begin{aligned}\frac{\partial}{\partial v_b}\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right) &= \frac{\frac{\partial}{\partial v_b}\left(B'_{z_r}R'_{v_b} - B'_{v_b}R'_{z_r} - B'_{z_r}R'_{v_r} + B'_{v_r}R'_{z_r}\right)}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}} \\ &= -\frac{\left(\frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{z_r(t^*)} + \left(2 - z_r\left(t^*\right)\right)e^{-z_r(t^*)}\right)\frac{z_r(0)}{\left(1-e^{-z_r(t^*)}+z_r(t^*)e^{-z_r(t^*)}\right)^2}\frac{\Psi^2}{(v_b)^2}}{\frac{z_b(t^*)e^{-z_b(t^*)}}{z_b(0)} - \frac{1}{a-1}\frac{1-e^{-z_b(t^*)}}{z_b(0)}} \\ &\quad - \frac{\left(\frac{a}{a-1} - 2\right)\frac{1}{(v_b)^2}\Psi}{\frac{z_b(t^*)e^{-z_b(t^*)}}{z_b(0)} - \frac{1}{a-1}\frac{1-e^{-z_b(t^*)}}{z_b(0)}},\end{aligned} \tag{58}$$

$$\begin{aligned}\frac{\partial}{\partial p_b}\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right) &= \frac{\frac{\partial}{\partial p_b}\left(B'_{z_r}R'_{p_b} - B'_{p_b}R'_{z_r} - B'_{z_r}R'_{p_r} + B'_{p_r}R'_{z_r}\right)}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}} \\ &= -\frac{\left(\frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{z_r(t^*)} + \left(2 - z_r\left(t^*\right)\right)e^{-z_r(t^*)}\right)\frac{z_r(0)}{\left(1-e^{-z_r(t^*)}+z_r(t^*)e^{-z_r(t^*)}\right)^2}\frac{\Psi^2}{(p_b)^2}}{\frac{z_b(t^*)e^{-z_b(t^*)}}{z_b(0)} - \frac{1}{a-1}\frac{1-e^{-z_b(t^*)}}{z_b(0)}} \\ &\quad - \frac{\frac{a}{a-1}\left(1 - 2\frac{1-e^{-z_r(t^*)}}{1-e^{-z_r(t^*)}+z_r(t^*)e^{-z_r(t^*)}}\right)\frac{1}{(p_b)^2}\Psi}{\frac{z_b(t^*)e^{-z_b(t^*)}}{z_b(0)} - \frac{1}{a-1}\frac{1-e^{-z_b(t^*)}}{z_b(0)}},\end{aligned} \tag{59}$$

$$\begin{aligned}\frac{\partial}{\partial p_b}\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right) &= \frac{\frac{\partial}{\partial p_b}\left(B'_{z_r}R'_{v_b} - B'_{v_b}R'_{z_r} - B'_{z_r}R'_{v_r} + B'_{v_r}R'_{z_r}\right)}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}} \\ &= \frac{\left(\frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{z_r(t^*)} + \left(2 - z_r\left(t^*\right)\right)e^{-z_r(t^*)}\right)\frac{z_r(0)}{\left(1-e^{-z_r(t^*)}+z_r(t^*)e^{-z_r(t^*)}\right)^2}\frac{1}{p_b}\frac{1}{v_b}\Psi^2}{\frac{z_b(t^*)e^{-z_b(t^*)}}{z_b(0)} - \frac{1}{a-1}\frac{1-e^{-z_b(t^*)}}{z_b(0)}} \\ &\quad + \frac{\frac{1}{a-1}\frac{1}{p_b}\frac{1}{v_b}\Psi - \frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{1-e^{-z_r(t^*)}+z_r(t^*)e^{-z_r(t^*)}}\frac{1}{p_b}\frac{1}{v_b}\Psi}{\frac{z_b(t^*)e^{-z_b(t^*)}}{z_b(0)} - \frac{1}{a-1}\frac{1-e^{-z_b(t^*)}}{z_b(0)}},\end{aligned} \tag{60}$$

$$\begin{aligned}\frac{\partial}{\partial v_b}\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right) &= \frac{\frac{\partial}{\partial v_b}\left(B'_{z_r}R'_{p_b} - B'_{p_b}R'_{z_r} - B'_{z_r}R'_{p_r} + B'_{p_r}R'_{z_r}\right)}{B'_{z_b}R'_{z_r} - B'_{z_r}R'_{z_b}} \\ &= \frac{\left(\frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{z_r(t^*)} + \left(2 - z_r\left(t^*\right)\right)e^{-z_r(t^*)}\right)\frac{z_r(0)}{\left(1-e^{-z_r(t^*)}+z_r(t^*)e^{-z_r(t^*)}\right)^2}\frac{1}{p_b}\frac{1}{v_b}\Psi^2}{\frac{z_b(t^*)e^{-z_b(t^*)}}{z_b(0)} - \frac{1}{a-1}\frac{1-e^{-z_b(t^*)}}{z_b(0)}} \\ &\quad + \frac{\frac{1}{a-1}\frac{1}{p_b}\frac{1}{v_b}\Psi - \frac{a}{a-1}\frac{1-e^{-z_r(t^*)}}{1-e^{-z_r(t^*)}+z_r(t^*)e^{-z_r(t^*)}}\frac{1}{p_b}\frac{1}{v_b}\Psi}{\frac{z_b(t^*)e^{-z_b(t^*)}}{z_b(0)} - \frac{1}{a-1}\frac{1-e^{-z_b(t^*)}}{z_b(0)}}.\end{aligned} \tag{61}$$

Substituting $\left(\frac{dz_b}{dv_b} - \frac{dz_b}{dv_r}\right)$ and $\left(\frac{dz_b}{dp_b} - \frac{dz_b}{dp_r}\right)$ and their derivatives using equations (51), (52) and (58) to (61) into the second derivatives given by equations (42) to (45) and rearranging implies,

$$\frac{\partial^2 \mu(G)}{\partial (v_b)^2} = -ua\frac{\left(z_r(t^*)\,e^{-z_r(t^*)}\right)^2 + z_r(t^*)\left(1-e^{-z_r(t^*)}\right)}{\left(1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}\right)^2}\frac{1}{(v_b)^2}\Psi < 0,$$

$$\frac{\partial^2 \mu(G)}{\partial (p_b)^2} = -ua\frac{\left(z_r(t^*)\,e^{-z_r(t^*)}\right)^2 + z_r(t^*)\left(1-e^{-z_r(t^*)}\right)}{1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}}\frac{1}{(p_b)^2}\Psi < 0,$$

$$\frac{\partial^2 \mu(G)}{\partial v_b \partial p_b} = ua\frac{\left(z_r(t^*)\,e^{-z_r(t^*)}\right)^2 + z_r(t^*)\left(1-e^{-z_r(t^*)}\right)}{\left(1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}\right)^2}\frac{1}{p_b}\frac{1}{v_b}\Psi > 0,$$

$$\frac{\partial^2 \mu(G)}{\partial p_b \partial v_b} = ua\frac{\left(z_r(t^*)\,e^{-z_r(t^*)}\right)^2 + z_r(t^*)\left(1-e^{-z_r(t^*)}\right)}{\left(1-e^{-z_r(t^*)} + z_r(t^*)\,e^{-z_r(t^*)}\right)^2}\frac{1}{p_b}\frac{1}{v_b}\Psi > 0.$$

This implies that the Hessian matrix is negative semidefinite, since

$$\frac{\partial^2 \mu(G)}{\partial (v_b)^2}\frac{\partial^2 \mu(G)}{\partial (p_b)^2} - \frac{\partial^2 \mu(G)}{\partial v_b \partial p_b}\frac{\partial^2 \mu(G)}{\partial p_b \partial v_b} = 0.$$

B Decomposition theorem and algorithm

Decomposition Theorem (Corominas-Bosch, 2004):

(1) Every graph G can be decomposed into a number of firm subgraphs (G_1^f,..., $G_{n_f}^f$), worker subgraphs (G_1^w,..., G_{nw}^w) and even subgraphs (G_1^e,..., G_{ne}^e) in such a way that each node (firm or worker) belongs to one and only one subgraph and any firm (worker) in a firm-(worker-) subgraph $G_{i,}^f(G_{i,}^w)$ is only linked to workers (firms) in a firm-(worker-)subgraph $G_j^f(G_j^w)$.
(2) Moreover, a given node (firm or worker) always belongs to the same type of subgraph for any such decomposition. We will write $G = G_1^f \cup ... \cup G_{n_f}^f \cup G_1^w \cup ... \cup G_{nw}^w \cup G_1^e \cup ... \cup G_{ne}^e$, with the union being disjoint.

The decomposition algorithm of Corominas-Bosch (2004) works as follows:
Step a: Eliminate all vacancies that did not receive any applicants.
Step b: For $k = 2, ..., v$, identify the groups of k vacancies that are jointly linked to less than k workers. Remove and collect them. We refer to those subgraphs as firm subgraphs.
Step c: Repeat step b but now reverse the role of workers and vacancies. The resulting subgraphs are called worker subgraphs.
Step d: When all worker subgraphs are removed, the remaining ones are balanced (or even) subgraphs (with an equal number of workers and firms).